ON A GENERAL CLASS OF SERIES OF THE FORM $\sum c_n g(x+n)$*

BY

R. D. CARMICHAEL

INTRODUCTION

In several important investigations, beginning with two memoirs† by Poincaré in 1885 and 1886 respectively, functions have arisen which are analytic in a sector of the complex plane and in that sector have a determinate asymptotic character (in the sense of the definition of Poincaré). In these investigations it has been usual to represent the asymptotic character of such a function by means of a divergent power series and to find in addition a convergent expansion which puts in evidence the analytic character of the function in the sector. Often this convergent expansion is complicated in character.

Obviously, it is desirable, if possible, to have a convergent expansion which accomplishes the latter object and at the same time readily shows the asymptotic character of the function. Recently Nörlund‡ has shown that functions which are characterized by being analytic in a half-plane and having an asymptotic character of a certain broad type are expansible in convergent factorial series and that these series do in fact readily show the asymptotic character of the functions represented by them. He has thus made manifest the fundamental importance of this class of series in investigating the properties of a function in the neighborhood of a singularity of a certain type.

Some months ago I observed that the factorial series is but an instance of a large class§ of series of simple properties and suitable for the representation of functions which are defined throughout a half-plane and have certain types of singularities at infinity. These series are of the form

* Presented to the Society (at Columbus), December 31, 1915.

† American Journal of Mathematics, vol. 7 (1885), pp. 203–258; Acta Mathematica, vol. 8 (1886), pp. 295–344. Particular instances of such functions appeared earlier, notably in the work of Stirling and Cauchy (see Borel's *Séries divergentes*, p. 30).

‡ Acta Mathematica, vol. 37 (1914), pp. 327–387. Two other valuable contributions leading in the same general direction should be mentioned, namely, a paper by Horn in Mathematische Annalen, vol. 71 (1912), pp. 510–532 and one by Watson in Rendiconti del Circolo Matematico di Palermo, vol. 34 (1912), pp. 41–88.

§ In § 6 it is seen incidentally that the power series is also essentially contained in this class.

$$\overline{\Omega}(x) = \sum_{n=0}^{\infty} c_n \frac{g(x+n)}{g(x)},$$

where $g(x)$ is a given function of x and the coefficients c_n are independent of x. With these series I associate others of the form

$$\Omega(x) = \sum_{n=0}^{\infty} c_n g(x+n).$$

More precise definitions are given at the beginning of § 1. As will be seen from these definitions, the fundamental characteristic of $g(x)$ is its asymptotic behavior. A detailed study of the more important series included in these classes and especially of their suitability for the representation of certain types of functions is desirable and will doubtless have important consequences in the development of the theory of functions. Results of this character I expect to communicate to the Society in later contributions. From these latter the important connection of the series $\overline{\Omega}(x)$ with the function-theoretic considerations mentioned in the first two paragraphs of this introduction will become apparent. In the present memoir I lay the foundations of the general theory of these series. Among the results obtained there is a set which corresponds closely, in point of completeness, to those contained in Landau's valuable memoir* on the foundations of the theory of factorial series. In the derivation of these results I have found Landau's methods of constant value and have employed them freely.

After a definition in § 1 of the classes $\Omega(x)$ and $\overline{\Omega}(x)$ of series in consideration I proceed in §§ 2–4 to the complete determination of the *character* of the regions of convergence and absolute convergence of the series and to a consideration of the character of their convergence in respect to uniformity. Certain auxiliary series (of the nature of generalized Dirichlet series) are introduced in § 5 and their properties are developed so far as they are needed for the determination in § 6 of the precise regions of convergence and absolute convergence of a particular series $\Omega(x)$ or $\overline{\Omega}(x)$ in terms of its coefficients c_n. In § 7 further relations between the series $\Omega(x)$ and the auxiliary generalized Dirichlet series are developed and are then employed in showing the existence, in certain cases, of singularities of functions defined by the series $\Omega(x)$. In § 8 I observe that a function can sometimes be expanded in a series $\Omega(x)$ or a series $\overline{\Omega}(x)$ dependent on a given function $g(x)$, in more than one way and indeed even in an unlimited number of ways; but if a certain broad addi-

*Sitzungsberichte der mathematisch-physikalischen Klasse der königlichen Bayerischen Akademie der Wissenschaften zu München, vol. 36 (1906): 151–218. This paper is valuable both for its elegant exposition of the theory of factorial series and for its new methods and results and for the historical remarks contained in it.

tional condition is placed on the function $g(x)$ it is shown that the expansion of a function in the series $\Omega(x)$ or in the series $\overline{\Omega}(x)$ can be effected in no more than one way.

1. Definitions and general notions

Let $g(x)$ be a single-valued analytic function of the complex variable x having the asymptotic character

$$g(x) \sim x^{P(x)} e^{Q(x)} \left(1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots\right) \tag{1}$$

valid for x approaching infinity in a positive sense along any line whatever parallel to the axis of reals, the functions $P(x)$ and $Q(x)$ being polynomials which we write in the form

$$P(x) = \mu_0 + \mu_1 x + \mu_2 x^2 + \cdots + \mu_k x^k \qquad (\mu_k \neq 0 \text{ if } k > 0),$$

$$Q(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_m x^m \qquad (\alpha_m \neq 0 \text{ if } m > 0).$$

In case $k = 0$ we assume that m is greater than unity in order to avoid cases which are unimportant so far as our present objects are concerned.* By r^t we mean $e^{t \log r}$ where the principal determination of $\log r$ is taken.

The asymptotic relation (1) is to be understood as an abbreviation for the infinite sequence of limits

$$\lim x^s \left\{ g(x) x^{-P(x)} e^{-Q(x)} - \left(1 + \frac{a_1}{x} + \cdots + \frac{a_s}{x^s}\right)\right\} = 0 \tag{2}$$

$$(s = 0, 1, 2, \cdots),$$

the limit being taken for x approaching infinity in a positive sense along any line whatever parallel to the axis of reals.†

By means of any function $g(x)$ possessing the properties just indicated we introduce the series $\Omega(x)$ defined by the relation

$$\Omega(x) = \sum_{n=0}^{\infty} c_n g(x + n) \tag{3}$$

where the coefficients $c_0, c_1, c_2, \cdots$ are independent of x. For the applications it will often be convenient to consider the series $\overline{\Omega}(x)$ which we write in the form

* The case excluded not only is unimportant for our purposes but it also requires a different method of treatment.

† The reader, in the course of perusing the paper, will readily observe that a weaker restriction than that indicated by the infinite set of limits (2) will suffice for the derivation of many of the results. In particular, throughout §§ 2–4 (in which the character of the regions of convergence and absolute convergence of $\Omega(x)$ is determined) the limits for $s = 0$ and $s = 1$ will suffice; and that for $s = 0$ alone will suffice unless simultaneously $k = 1$ and $m = 0$ or 1.

$$\overline{\Omega}(x) \equiv \frac{\Omega(x)}{g(x)} = \sum_{n=0}^{\infty} c_n \frac{g(x+n)}{g(x)}. \tag{4}$$

A value of x for which any one of the functions $g(x)$, $g(x+1)$, $g(x+2)$, $\cdots$ $[g(x+1)/g(x), g(x+2)/g(x), \cdots]$ has a singularity will be called an *exceptional value* or an *exceptional point* for the series $\Omega(x)[\overline{\Omega}(x)]$. Other values of x will be called *non-exceptional*. In order to avoid cases without value for our purposes we shall assume that $g(x)$ is of such character that there is a (two-dimensional) region in the x-plane containing no exceptional points for the series.

That a large and important class of series $\Omega(x)$ and $\overline{\Omega}(x)$ exist in accordance with the foregoing definition is readily apparent from the following considerations.

In the first place there exists an extensive and easily accessible class of functions $g(x)$. In particular, such functions are introduced to our notice by the linear homogeneous differential equation

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + P_n y = 0,$$

in which the coefficients $P_0, P_1, \cdots, P_n$ are polynomials in x subject to the condition that their degrees do not constantly decrease as one passes in order from the first to the last polynomial in the sequence. Poincaré* has in fact shown that a solution $y(x)$ of such an equation is asymptotic to an expression of the form of the second member of (1) when the polynomial $P(x)$ is taken to be a constant and x approaches infinity in an appropriate direction. Hence a constant c exists such that $y(cx)$ is a function having the property specified above for $g(x)$.

A still more effective (because simpler) class of functions $g(x)$ is afforded by linear homogeneous difference equations whose coefficients have at infinity the character of rational functions and whose associated characteristic algebraic equation has its roots different from each other and from zero. Such an equation has a fundamental set of solutions† each function of which is of the character specified above for $g(x)$, the polynomials $P(x)$ and $Q(x)$ in this case being of the first degree. The series $\overline{\Omega}(x)$ has been treated in detail (see Landau, l. c.) for the two special values‡ $g(x) = \Gamma(x)$ and $g(x) = 1/\Gamma(x)$ belonging to this sub-class. The latter value of $g(x)$ gives rise to the simple factorial series

*Acta Mathematica, vol. 8 (1886), pp. 295–344.

†Birkhoff, these Transactions, vol. 12 (1911): 242–284. See also an earlier paper of mine in the same volume.

‡For the first case mentioned a simple transformation is necessary in order to throw the series $\overline{\Omega}(x)$ into the form treated by Landau.

$$\overline{\Omega}(x) = c_0 + \sum_{n=0}^{\infty} \frac{c_{n+1}}{x(x+1)\cdots(x+n)},$$

the importance of which has already been indicated in the introduction.*

Among the important special types of series of the class $\overline{\Omega}(x)$ introduced by means of difference equations is the important generalization of factorial series afforded by the series

$$\overline{\Omega}(x) = c_0 + \sum_{n=0}^{\infty} c_{n+1} A(x) A(x+1) \cdots A(x+n), \tag{5}$$

where $A(x)$ is any rational function of x having a zero value for $x = \infty$. It arises through taking for $g(x)$ an appropriate solution (in fact, the so-called first principal solution) of the equation $f(x+1) = A(x)f(x)$; for from this equation we have readily

$$f(x+n+1) = A(x+n)A(x+n-1)\cdots A(x+1)A(x)f(x).$$

In determining the character of the region of convergence of $\Omega(x)$ (and consequently of $\overline{\Omega}(x)$) we have need for two lemmas† which it is convenient to state here as follows:

LEMMA I. *Let $u_1 + u_2 + u_3 + \cdots$ be a convergent series and let $v_1, v_2, v_3, \cdots$ be an infinite sequence of numbers such that the series*

$$\sum_{n=1}^{\infty} |v_n - v_{n+1}|$$

is convergent. Then the series

$$u_1 v_1 + u_2 v_2 + u_3 v_3 + \cdots$$

is convergent.

LEMMA II. *Let $u_1 + u_2 + u_3 + \cdots$ be a convergent series of constant terms and let $v_1, v_2, v_3, \cdots$ be an infinite sequence of functions of the complex variable x analytic in a given closed domain D and such that the series*

$$\sum_{n=1}^{\infty} |v_n - v_{n+1}|$$

converges uniformly in D. Then the series

$$u_1 v_1 + u_2 v_2 + u_3 v_3 + \cdots$$

converges uniformly in D.

* The definitions of the series $\Omega(x)$ and $\overline{\Omega}(x)$ might be extended to the case in which the series in (1) proceeds in certain descending fractional powers of x. The resulting series are amenable to the general methods of this paper. Again, suitable functions $g(x)$ for these series are readily afforded by the solutions of linear differential and difference equations.

† These lemmas have been employed by several writers. The reader will conveniently find proofs of them (in a generalized form) in the Bulletin of the American Mathematical Society, vol. 20, pp. 225–233. Historical remarks concerning their previous use will be found in Landau's memoir (already cited).

2. Character of the region of convergence

Suppose that the series $\Omega(x)$ converges for a given value x_0 of x which is non-exceptional for $\Omega(x)$. We shall find conditions on a value x_1 of x which are sufficient to ensure that $\Omega(x)$ converges for $x = x_1$, x_1 itself being a non-exceptional value for $\Omega(x)$. For this purpose we may employ Lemma I, taking

$$u_n = c_n\, g(x_0 + n), \qquad v_n = \frac{g(x_1 + n)}{g(x_0 + n)}, \qquad n \geqq l;$$

here the integer l is chosen so that $g(x_0 + n) \neq 0$ if $n \geqq l$. Then $\Omega(x_1)$ will converge* provided the series

$$\sum_{n=l}^{\infty} |v_n - v_{n+1}| \tag{6}$$

is convergent.

For determining when the series (6) converges we make use of the asymptotic character of v_n with respect to n. To obtain a formula for this it is convenient to write first the formula for the asymptotic character of $g(x + n)$ with respect to n when x is held fixed. We have

$$g(x + n) \sim e^{p_n(x) \log n + q_n(x)}\, s_n(1) \tag{7}$$

where $s_n(\alpha)$ denotes a series in non-negative integral powers of $1/n$ with constant term α and $p_n(x)$ and $q_n(x)$ are quantities which may be written in the form

$$\begin{aligned} p_n(x) &= \nu_{0n} + \nu_{1n}\, x + \nu_{2n}\, x^2 + \cdots + \nu_{kn}\, x^k, \\ q_n(x) &= \beta_{0n} + \beta_{1n}\, x + \beta_{2n}\, x^2 + \cdots + \beta_{mn}\, x^m \\ &\qquad + \gamma_{1n}\, x + \gamma_{2n}\, x^2 + \cdots + \gamma_{kn}\, x^k. \end{aligned} \tag{8}$$

Here the quantities ν, β, γ have the values

$$\begin{aligned} \nu_{sn} &= \sum_{t=s}^{k} \binom{t}{s} \mu_t\, n^{t-s}, \qquad s = 0, 1, \cdots, k; \\ \beta_{sn} &= \sum_{t=s}^{m} \binom{t}{s} \alpha_t\, n^{t-s}, \qquad s = 0, 1, \cdots, m; \\ \gamma_{sn} &= \sum_{t=0}^{k} \mu_t\, n^{t-s} \left[\binom{t}{s-1} - \frac{1}{2}\binom{t}{s-2} + \frac{1}{3}\binom{t}{s-3} - \cdots \right] \\ &\qquad (s = 1, 2, \cdots, k). \end{aligned} \tag{9}$$

From (7) it follows that

$$v_n \sim e^{\log n \cdot \{p_n(x_1) - p_n(x_0)\} + \{q_n(x_1) - q_n(x_0)\}}\, s_n(1). \tag{10}$$

* We employ the statement $\Omega(x_1)$ *converges* as an abbreviation for the statement *the series* $\Omega(x)$ *converges for the value* x_1 *of* x.

By means of this relation we determine conditions under which series (6) is convergent. We see that $p_n(x_1) - p_n(x_0)$ is a polynomial in n of degree $k-1$ and that $q_n(x_1) - q_n(x_0)$ is a polynomial in n and that its degree is $m-1$ in case m is greater than k. (In case k or m is zero the corresponding difference function is zero.) There is obviously a separation of cases according as k is or is not less than m.

Suppose first that $k \geqq m$. Then it is clear that the dominating term in the exponent of the second member of (10), when this exponent is expressed as a polynomial in n and $\log n$, is

$$k\mu_k(x_1 - x_0)\, n^{k-1} \log n.$$

Then it is easy to see that series (6) converges if $R\{\mu_k(x_1 - x_0)\}$* is less than zero, since clearly it is dominated term by term by a convergent series. This is obvious in case $k > 1$. If $k = 1$ we have

$$v_n \sim n^{\mu_1(x_1 - x_0)}\left(t_0 + \frac{t_1}{n} + \cdots\right),$$

where $t_0, t_1, \cdots$ are determinate constants. Then

$$v_n - v_{n+1} \sim n^{\mu_1(x_1 - x_0)-1}\left(t_0\,\mu_1(x_0 - x_1) + \frac{\xi}{n} + \cdots\right),$$

where ξ is a determinate constant. From this it follows in this case also that series (6) converges if $R\{\mu_k(x_1 - x_0)\}$ is negative.

Next suppose that $k < m$. Then $m \geqq 2$, since by hypothesis $m > 1$ if $k = 0$. The dominating term in the exponent in the second member of (10), when this exponent is expressed as a polynomial in n and $\log n$, is

$$m\alpha_m(x_1 - x_0)\, n^{m-1}.$$

Hence, in this case series (6) converges when $R\{\alpha_m(x_1 - x_0)\}$ is negative.

From Lemma I it follows that the convergence of (6) is sufficient to ensure the convergence of $\Omega(x_1)$. Hence we are led to the following theorem:

THEOREM I. *Let x_0 and x_1 be two values of x which are non-exceptional for the series $\Omega(x)$ and suppose that $\Omega(x_0)$ converges. Then*

(1) *in case $k \geqq m$, $\Omega(x_1)$ converges if $R(\mu_k x_1) < R(\mu_k x_0)$;*

(2) *in case $k < m$, $\Omega(x_1)$ converges if $R(\alpha_m x_1) < R(\alpha_m x_0)$.*

By the *region C of convergence* of the series $\Omega(x)$ we shall mean a region such that $\Omega(x)$ converges for every non-exceptional value of x in the interior of C and diverges for every non-exceptional value of x exterior to C. We shall likewise call C the region of convergence of the series $\overline{\Omega}(x)$ since ob-

* The symbol $R(z)$ is used to denote the real part of z.

viously $\Omega(x)$ and $\overline{\Omega}(x)$ both converge or both diverge for a given non-exceptional value of x for which $g(x)$ is finite and different from zero.

By means of the foregoing theorem it is easy to determine the nature of the region of convergence of $\Omega(x)$. Through examples it may be shown in the first place that particular series $\Omega(x)$ exist which diverge for every non-exceptional value of x and that there are other particular series $\Omega(x)$ which converge for every non-exceptional value of x. In fact such examples are readily afforded by the theory of factorial series. We shall assume for the moment that we are dealing with a series $\Omega(x)$ which belongs neither to the class of everywhere convergent series nor to the class of everywhere divergent series. Let σ denote α_m or μ_k according as m is or is not greater than k. Consider the real numbers $R(\sigma x)$ for varying values of x which are non-exceptional for $\Omega(x)$ and divide them into two classes A and B according as x is a point of convergence or a point of divergence for $\Omega(x)$. From Theorem I it follows that no B is less than any A. Then it is clear that a number λ exists such that $\Omega(x)$ converges [diverges] for every non-exceptional value of x such that $R(\sigma x) < \lambda\,[R(\sigma x) > \lambda]$. Such a number λ we shall call a convergence number for $\Omega(x)$. It may conveniently be said that $-\infty\,[+\infty]$ is a convergence number for $\Omega(x)$ when $\Omega(x)$ diverges [converges] for all non-exceptional values of x.

It is clear that a series $\Omega(x)$ has a unique convergence number λ if the non-exceptional values of x are such that the numbers $R(\sigma x)$ form a dense set on the whole of the axis of reals from $-\infty$ to $+\infty$; and this in fact is the case in the applications of the theory which we shall have occasion to make. When there is but one convergence number λ we shall call it *the* convergence number for $\Omega(x)$. If there is more than one convergence number λ then for *the* convergence number for $\Omega(x)$ we shall take the least convergence number or the greatest lower bound to the convergence numbers in case a least one does not exist.

If λ is the convergence number for $\Omega(x)$, then the equation $R(\sigma x) = \lambda$ represents a straight line which is clearly the boundary of the region of convergence of $\Omega(x)$, this region being that for which $R(\sigma x) < \lambda$.

The results just derived may be stated in the following theorem:

THEOREM II. *If σ denotes α_m or μ_k according as m is or is not greater than k, then there exists a unique real number λ such that the region of convergence of the series $\Omega(x)\,[\overline{\Omega}(x)]$ is bounded by the straight line $R(\sigma x) = \lambda$ and lies on that side of this line for which $R(\sigma x) < \lambda$.*

For particular functions $g(x)$ the series $\Omega(x)$ may converge everywhere on the boundary line l of its region of convergence, or it may converge nowhere on l, or it may converge at some points on l and diverge at others, as one may see readily from the special case of factorial series; and these possibilities

may severally be realized through the character of the coefficients c_n. But there are certain general classes of cases in which the question as to convergence on l is answered (except for a finite number of points) by the character of the function $g(x)$ itself. For the discussion of these it is convenient to distinguish four cases.

Suppose first that k is greater than either 1 or m, or that $k = m > 1$ and μ_k and α_k lie on the same straight line through the point zero in the complex plane. Let us consider the question of convergence of series (6) when

$$R(\mu_k x_1) = R(\mu_k x_0).$$

This depends on the asymptotic character of v_n as given in (10). It is easy to see that (6) converges provided that

$$R\{\tfrac{1}{2}k\mu_k(x_1^2 - x_0^2) + \mu_{k-1}(x_1 - x_0)\} < 0.$$

By an argument similar to that by which Theorem II was proved it may now be shown that if $\Omega(x)$ converges for any non-exceptional value x_0 of x on the line l then there exists a unique real number λ_1 such that $\Omega(x_1)$ converges if $R_1 < \lambda_1$ and diverges if $R_1 > \lambda_1$, R_1 being defined by the relation

$$R_1 = R\{\tfrac{1}{2}k\mu_k x_1^2 + \mu_{k-1} x_1\}$$

and x_1 being a non-exceptional value of x. In this case the only non-exceptional points x_1 at which the question of convergence of $\Omega(x)$ is unanswered are those for which we have simultaneously $R_1 = \lambda_1$ and $R(\mu_k x_1) = \lambda$; and clearly these are at most two in number.

If $k = m > 1$ and μ_k and α_k do not lie on the same straight line through the point 0 in the complex plane it may be shown in a similar manner that if $\Omega(x)$ converges for a non-exceptional value x_0 on the line l then a unique real number λ_1 exists such that $\Omega(x_1)$, where x_1 is non-exceptional, converges or diverges according as $R\{x_1(\mu_k + \alpha_k)\}$ is less than λ_1 or greater than λ_1. In this case the only non-exceptional point x_1 at which the question of convergence of $\Omega(x_1)$ is unanswered is that for which we have simultaneously

$$R(\mu_k x_1) = \lambda, \qquad R\{x_1(\mu_k + \alpha_k)\} = \lambda_1.$$

We suppose next that $m = k + 1$ and that μ_k and α_{k+1} do not lie on the same straight line through the point 0 of the complex plane. Then $k \geqq 1$ since $m > 1$ if $k = 0$. Proceeding as before we find that if $\Omega(x)$ converges for a non-exceptional value x_0 on the line l then a unique real number λ_1 exists such that $\Omega(x_1)$, where x_1 is non-exceptional, converges or diverges according as $R(\mu_k x_1)$ is less than λ_1 or is greater than λ_1. In this case the only non-exceptional point x_1 at which the question of convergence of $\Omega(x_1)$ is unanswered is that for which we have simultaneously

$$R(\alpha_{k+1} x_1) = \lambda, \qquad R(\mu_k x_1) = \lambda_1.$$

Finally, suppose that $m > k+1$, or that $m = k+1$ and μ_k and α_{k+1} lie on the same straight line through the point 0 of the complex plane. In this case we find that if $\Omega(x)$ converges for a non-exceptional value x_0 of x on the line l then a unique real number λ_1 exists such that $\Omega(x_1)$ converges if $R_1 < \lambda_1$ and diverges if $R_1 > \lambda_1$, R_1 being defined by the relation

$$R_1 = R(\tfrac{1}{2} m \alpha_m x_1^2 + \alpha_{m-1} x_1)$$

and x_1 being a non-exceptional value of x. In this case the only non-exceptional points x_1 at which the question of convergence of $\Omega(x_1)$ is unanswered are those for which we have simultaneously

$$R_1 = \lambda_1, \qquad R(\alpha_m x_1) = \lambda.$$

It is obvious that certain other cases, exceptional in character, may be similarly treated by the general method employed in the four preceding paragraphs.

3. Uniform convergence

Turning now to the question of uniform convergence, we shall prove the following theorem:

Theorem III. *The series $\Omega(x)[\overline{\Omega}(x)]$ converges uniformly in any closed domain D which lies within the region of convergence of $\Omega(x)[\overline{\Omega}(x)]$ and contains no points which are exceptional for this series or are limit points of points which are exceptional for the series.*

It is sufficient to prove the theorem for the case of the series $\Omega(x)$; for the other series may obviously be treated in a similar manner.

From the character of D as specified in the theorem it is clear that a non-exceptional point x_0 exists such that $\Omega(x_0)$ converges and $R(\sigma x_0)$ is less than $R(\sigma x_1)$ for every value of x_1 lying in D, σ being used as before for α_m or μ_k according as m is or is not greater than k. From Lemma II it follows that for the proof of Theorem III it is sufficient to show that series (6) converges uniformly for x_1 in D and x_0 fixed as just indicated. For the proof of the latter we employ relation (10), which is valid for every value of x_1 in D. By its aid one may construct, in an obvious manner, a series of positive constant terms which is term by term greater than series (6) for every x_1 in D; whence, through use of a theorem of Weierstrass, it follows that (6), and hence $\Omega(x_1)$, is uniformly convergent in D.

As an immediate consequence of Theorem III we see that *the sum of the series $\Omega(x)[\overline{\Omega}(x)]$ is a function $S(x)$ of x which is analytic at every non-exceptional point which is in the interior of its region of convergence and is not a*

limit point of exceptional points, and the derivatives of $S(x)$ at every such point may be found by differentiating the series $\Omega(x)[\overline{\Omega}(x)]$ term by term.

4. Character of the region of absolute convergence

In order to determine the character of the region of absolute convergence of $\Omega(x)$ let us consider a non-exceptional point x_0 for which $\Omega(x_0)$ converges absolutely. Then it is obvious that $\Omega(x_1)$ will converge absolutely provided that x_1 is non-exceptional for $\Omega(x)$ and $|v_n|$ is bounded, v_n being defined as before in terms of x_1 and x_0. Again we have the asymptotic relation (10). From this it follows readily that $|v_n|$ is bounded provided that

$$R(\sigma x_1) < R(\sigma x_0).$$

Hence we have the following theorem:

Theorem IV. *Let σ denote α_m or μ_k according as m is or is not greater than k. Let x_0 and x_1 be two non-exceptional values of x for the series $\Omega(x)$ and suppose that $\Omega(x_0)$ converges absolutely. Then $\Omega(x_1)$ will converge absolutely if*

$$R(\sigma x_1) < R(\sigma x_0).$$

By *the region Γ of absolute convergence* of the series $\Omega(x)$ we shall mean a region such that $\Omega(x)$ converges absolutely for every non-exceptional value of x in the interior of Γ and fails to converge absolutely for each non-exceptional value of x exterior to Γ. We shall also call Γ the region of absolute convergence for the series $\overline{\Omega}(x)$, since obviously $\Omega(x)$ and $\overline{\Omega}(x)$ both converge absolutely or both fail to converge absolutely for a given non-exceptional value of x for which $g(x)$ is finite and different from 0.

The nature of this region Γ may readily be determined by means of Theorem IV. In fact, a discussion similar to that which leads up to Theorem II now yields the following result:

Theorem V. *If σ denotes α_m or μ_k according as m is or is not greater than k, then there exists a unique real number μ such that the region of absolute convergence of the series $\Omega(x)[\overline{\Omega}(x)]$ is bounded by the straight line $R(\sigma x) = \mu$ and lies on that side of this line for which $R(\sigma x) < \mu$.*

In certain general classes of cases the question concerning the absolute convergence of $\Omega(x)$ on the boundary line l of absolute convergence (except at a finite number of points) depends on the character of $g(x)$. A treatment of this matter similar to that following Theorem II for the case of the boundary line of convergence of $\Omega(x)$ yields a set of conclusions precisely similar to those obtained for the latter problem.

5. Properties of a certain class of auxiliary series

In view of the asymptotic form of $g(x+n)$ with respect to n when x is held fixed, as given in relation (7), it is natural to introduce* in connection with $\Omega(x)$ the generalized† Dirichlet series‡

$$D(x) = \sum_{n=1}^{\infty} c_n e^{\lambda_{1n}x + \lambda_{2n}x^2 + \cdots + \lambda_{tn}x^t} \tag{11}$$

where the functions λ_{jn} of n, for $j = 1, 2, \cdots, t$, have the following properties:

(1) one at least of the functions λ_{jn} approaches infinity as n increases indefinitely;

(2) the limit $\lim_{n=\infty} \arg \lambda_{jn}$ exists for every j and is finite;

(3) in case t is greater than unity, one of the functions λ_{jn}, say λ_{sn} (where s is fixed), dominates all the others in such wise that

$$\lim_{n=\infty} \frac{\lambda_{jn}}{\lambda_{sn}} = 0, \qquad j \neq s;$$

(4) in case more than one function of the set λ_{jn}, $j = 1, 2, \cdots, t$, is not identically zero the function λ_{sn} is of such character that the series

$$\sum_{n=1}^{\infty} e^{-|\lambda_{sn}|\eta}$$

converges for every positive number η;

(5) in case λ_{sn} is the only function of the set λ_{jn}, $j = 1, 2, \cdots, t$, which is not identically zero then λ_{sn} either has the property named in (4) or $\arg \lambda_{sn}$ is a constant.

We shall denote the value of the limits in (2) by θ_j, $j = 1, 2, \cdots, t$, so that we have

$$\theta_j = \lim_{n=\infty} \arg \lambda_{jn} \qquad (j = 1, 2, \cdots, t).$$

Now let x_0 be a value of x such that $D(x_0)$ converges and consider what conditions may be placed on x_1 so that $D(x_1)$ shall certainly converge. Em-

* Compare the special case $\Delta(x)$ of $D(x)$ introduced at the beginning of § 6.

† The series commonly known as the generalized Dirichlet series is that special case of the series $D(x)$ in (11) for which $t = 1$ and λ_{1n} is real. I have not been able to find a discussion of the more general series $D(x)$ defined in the text.

‡ The properties of these generalized Dirichlet series are here developed only so far as they are needed in deriving the properties of the series $\Omega(x)$. It is clear that the theory is readily capable of further extension. Moreover, the generalization might be carried further in the direction indicated by the present treatment and yet the essential elegance of the fundamental theorems be retained. But our purpose here is to employ these series merely as a tool for investigating the properties of the series $\Omega(x)$ and $\overline{\Omega}(x)$; and the treatment is restricted accordingly.

ploying Lemma I and proceeding as in the case of the corresponding problem for the series $\Omega(x)$ we see that $D(x_1)$ will converge provided that

$$\sum_{n=1}^{\infty} |w_{n+1} - w_n| \tag{12}$$

converges, where w_n has the value

$$w_n = e^{\lambda_{1n}(x_1 - x_0) + \lambda_{2n}(x_1^2 - x_0^2) + \cdots + \lambda_{tn}(x_1^t - x_0^t)}.$$

Now, let us write $\lambda_{sn} = |\lambda_{sn}| e^{i\theta_{sn}}$. Then, by hypothesis, θ_{sn} approaches the finite value θ_s as n increases indefinitely. If we suppose that x_1 is subject to the condition

$$R\{e^{i\theta_s}(x_1^s - x_0^s)\} < 0 \tag{13}$$

it is clear that a positive constant η and a positive integer N exist such that

$$|w_n| < e^{-|\lambda_{sn}|\eta}$$

when n is greater than N. From this and properties (4) and (5) of λ_{sn} it follows that (12) certainly converges unless every function $\lambda_{jn}, j = 1, 2, \cdots, t$, except λ_{sn}, is identically equal to zero and λ_{sn} is of constant modulus θ_s. For this latter case the convergence of (12) may be shown in a different way. We now have

$$w_{n+1} - w_n = e^{\lambda_{n+1}\tau} - e^{\lambda_n \tau},$$

where

$$\lambda_n = |\lambda_{sn}| \qquad \text{and} \qquad \tau = e^{i\theta_s}(x_1^s - x_0^s).$$

Hence

$$w_{n+1} - w_n = \frac{1}{\tau}\int_{\lambda_n}^{\lambda_{n+1}} e^{u\tau}\, du;$$

so that

$$|w_{n+1} - w_n| \leqq \frac{1}{|\tau|}\int_{\lambda_n}^{\lambda_{n+1}} e^{uR(\tau)}\, du.$$

The series of which the nth term is the second member of this relation is obviously convergent since $R(\tau) < 0$ on account of the condition on x_1 expressed in (13). Therefore, the series in (12) is convergent, as was to be proved.

From our lemma it follows now that $D(x_1)$ converges provided that x_1 satisfies inequality (13). This result may obviously be used as Theorem I was employed in § 2 in the proof of Theorem II and with a corresponding result. Thus we are led to the following theorem:

THEOREM VI. *If $D(x_0)$ converges then $D(x_1)$ will converge if*

$$R(e^{i\theta_s} x_1^s) < R(e^{i\theta_s} x_0^s).$$

There exists a unique real number λ such that the region of convergence of the

*series $D(x)$ is bounded by the curve** $R(e^{i\theta_s} x_0^s) = \lambda$ and lies on that side of this curve for which $R(e^{i\theta_s} x^s) < \lambda$.*

Corresponding to Theorems IV and V of the preceding section, we have the following theorem which is readily proved:

THEOREM VII. *If $D(x_0)$ converges absolutely then $D(x_1)$ will converge absolutely if*

$$R(e^{i\theta_s} x_1^s) < R(e^{i\theta_s} x_0^s).$$

There exists a unique real number μ such that the region of absolute convergence of $D(x)$ is bounded by the curve $R(e^{i\theta_s} x^s) = \mu$ and lies on that side of this curve for which $R(e^{i\theta_s} x^s) < \mu$.

The question as to convergence of $D(x)$ for points on the boundary curves of convergence can be answered (except at a finite number of points), in a large range of cases, in terms of the properties of the functions λ_{jn}, $j = 1, 2, \cdots, t$; but we have no present use for these results and consequently they will not be stated.

Employing Lemma II and proceeding by a method in all respects similar to that used in the proof of Theorem III one may demonstrate the following theorem:

THEOREM VIII. *The series $D(x)$ converges uniformly in any closed domain S lying within its region of convergence.*

Consequently, the sum of the series is a function which is analytic within the region of convergence of the series.

We shall now prove the following theorem:

THEOREM IX. *The boundary curve of the region of convergence [absolute convergence] is the same for the two series $D(x)$ and $D_s(x)$, where*

$$D_s(x) = \sum_{n=1}^{\infty} c_n e^{\lambda_{sn} x^s}.$$

It should be observed that the theorem says nothing about convergence on the boundary of the region of convergence [absolute convergence].

If $D(x)$ and $D_s(x)$ are not identical then at least two of the functions $\lambda_{1n}, \cdots, \lambda_{tn}$ in (11) are not identically zero. Hence, so far as there is anything in the theorem to be proved, we have a λ_{sn} possessing property (4). The proof of the theorem falls into two parts.

1. Let $\bar{x}$ be any point in the interior of the region of convergence of $D(x)$. We shall show that $\bar{x}$ is likewise in the interior of the region of convergence of $D_s(x)$.

If $R(e^{i\theta_s} x^s) = \lambda$ is the boundary of the region of convergence of $D(x)$, then $R(e^{i\theta_s} \bar{x}^s) < \lambda$. It is obvious thence that numbers x_0 and x_1 exist such that

* This curve is a straight line only if $s = 1$.

$$R(e^{i\theta_s}\bar{x}^s) < R(e^{i\theta_s}x_1^s) < R(e^{i\theta_s}x_0^s) < \lambda. \tag{14}$$

Then x_0 and x_1 are points of convergence of $D(x)$. If we apply Lemma I, taking for u_n and v_n the values

$$u_n = c_n e^{\lambda_{1n}x_0+\cdots+\lambda_{tn}x_0^t}, \qquad v_n = e^{\lambda_{sn}x_1^s-(\lambda_{1n}x_0+\cdots+\lambda_{tn}x_0^t)},$$

we see that $D_s(x_1)$ converges if $\sum|v_n - v_{n+1}|$ converges. That the latter series converges follows at once from the fact that λ_{sn} possesses property (4) and that a positive number η and a positive integer N exist such that

$$|v_n| < e^{-|\lambda_{sn}|\eta},$$

when n is greater than N. From the convergence of $D_s(x_1)$, the relation between x_1 and $\bar{x}$ in (14), and Theorem VI as applied to $D_s(x_1)$, it follows that $\bar{x}$ is in the interior of the region of convergence of $D_s(x)$.

2. If we suppose next that $\bar{x}$ is a point in the interior of the region of convergence of $D_s(x)$ it may be shown that it is likewise in the interior of the region of convergence of $D(x)$. For this it is obviously sufficient to apply Lemma I in the same way as in the preceding case, taking for u_n and v_n the values

$$u_n = c_n e^{\lambda_{sn}x_0^s}, \qquad v_n = e^{\lambda_{1n}x_1+\cdots+\lambda_{tn}x_1^t-\lambda_{sn}x_0^s},$$

where x_0 and x_1 satisfy relation (14), the curve $R(e^{i\theta_s}x^s) = \lambda$ being now the boundary of the region of convergence of $D_s(x)$.

From the conclusions of the two preceding paragraphs we see that Theorem IX is true in so far as it relates to the region of convergence. The part relating to the region of absolute convergence may be proved in the same way; or it may be proved more directly by a term-by-term comparison of the absolutely convergent series $D(x_0)$ and $D_s(x_1)$ in the first case and $D(x_1)$ and $D_s(x_0)$ in the second case, x_0 and x_1 being connected with an interior point $\bar{x}$ of absolute convergence by a relation of the form (14), the curve $R(e^{i\theta_s}\bar{x}^s) = \lambda$ being now the boundary of the region of absolute convergence.

In the next section we shall have use also for the following somewhat special theorem:

THEOREM X. *Let λ_{sn} be separated into two parts ρ_{sn} and σ_{sn} in such a way that $\lambda_{sn} = \rho_{sn} + \sigma_{sn}$ and*

$$\lim_{n=\infty}\frac{\sigma_{sn}}{\rho_{sn}} = 0, \qquad \lim_{n=\infty}\arg\rho_{sn} \text{ exists and is finite.} \tag{15}$$

Then the boundary curve of the region of convergence [absolute convergence] is the same for the series $D_s(x)$ and $\overline{D}_s(x)$, where

$$\overline{D}_s(x) = \sum_{n=1}^{\infty} c_n e^{\rho_{sn}x^s},$$

provided that λ_{sn} has the property (4).

We observe that the series $\overline{D}_s(x)$ is obviously a special case of the series $D(x)$.

The proof of the theorem falls into two parts.

1. Let $\bar{x}$ be any point in the interior of the region of convergence of $D_s(x)$. We shall show that $\bar{x}$ is likewise in the interior of the region of convergence of $\overline{D}_s(x)$.

If $R(e^{i\theta_s}x^s) = \lambda$ is the boundary of the region of convergence of $D_s(x)$, then $R(e^{i\theta_s}\bar{x}^s) < \lambda$. It is thence obvious that numbers x_0 and x_1 exist such that relation (14) is satisfied. Then x_0 and x_1 are points of convergence of $D_s(x)$. If we apply Lemma I, taking for u_n and v_n the values

$$u_n = c_n e^{\lambda_{sn}x_0^s}, \qquad v_n = e^{\rho_{sn}x_1^s - \lambda_{sn}x_0^s} = e^{\lambda_{sn}(x_1^s - x_0^s) - \sigma_{sn}x_1^s},$$

we see that $\overline{D}_s(x_1)$ converges if $\sum|v_{n+1} - v_n|$ converges. That the latter series converges follows readily from relation (14), the second form of the value of v_n, and the fact that λ_{sn} has the property (4).

From the convergence of $\overline{D}_s(x_1)$, the relation between x_1 and $\bar{x}$ in (14), and Theorem VI as applied to the series $\overline{D}_s(x)$, it follows that $\bar{x}$ is in the interior of the region of convergence of $\overline{D}_s(x)$.

2. If $\bar{x}$ is any point in the interior of the region of convergence of $\overline{D}_s(x)$ then it may be shown by a method similar to that employed in the previous case that $\bar{x}$ is likewise in the interior of the region of convergence of $D_s(x)$.

Thus we have established the part of the theorem which refers to the region of convergence. That referring to the region of absolute convergence may readily be established by the method indicated for the corresponding part of the preceding theorem.

6. Determination of the regions of convergence of the series $\Omega(x)$ and $\overline{\Omega}(x)$ in terms of their coefficients

Since the series $\Omega(x)$ and $\overline{\Omega}(x)$ have the same region of convergence and the same region of absolute convergence we shall confine our attention to one of them alone, namely, $\Omega(x)$, in investigating the problem of the determination of the boundary lines of the regions of convergence and absolute convergence in terms of the coefficients c_n.

An instance $\Delta(x)$ of the series $D(x)$ is intimately related to the series $\Omega(x)$ and is a useful auxiliary in the solution of the problem treated in this section. It may be defined by the relation

$$\Delta(x) = \sum_{n=1}^{\infty} c_n e^{\sigma_n} e^{p_n(x)\log n + q_n(x) - \sigma_n}. \tag{16}$$

where σ_n has the value

$$\sigma_n = \nu_{0n} \log n + \beta_{0n} \tag{17}$$

and $p_n(x)$, $q_n(x)$, ν_{0n}, and β_{0n} are the functions denoted by these symbols in § 2. It is easy to see that $\Delta(x)$ is that special case of $D(x)$ in which c_n is replaced by $c_n e^{\sigma_n}$ and the relation

$$\lambda_{1n} x + \lambda_{2n} x^2 + \cdots + \lambda_{tn} x^t = p_n(x) \log n + q_n(x) - \sigma_n$$

is satisfied identically in x for every value of n. The last relation determines a set of functions $\lambda_{1n}, \cdots, \lambda_{tn}$ satisfying the conditions imposed at the beginning of § 5.

The following theorem states an important relation between $\Omega(x)$ and $\Delta(x)$:

THEOREM XI. *Let x_0 be a non-exceptional point for $\Omega(x)$. Then if either of the series $\Omega(x)$, $\Delta(x)$ converges [converges absolutely] for $x = x_0$ the other likewise converges [converges absolutely].*

The part of the theorem relating to absolute convergence is obvious in view of the asymptotic relation (7). It remains therefore to prove only the part relating to convergence.

Suppose first that $\Omega(x_0)$ converges. Put

$$u_n = c_n g(x_0 + n), \qquad v_n = \frac{e^{p_n(x_0) \log n + q_n(x_0)}}{g(x_0 + n)}.$$

Then from Lemma I it follows that $\Delta(x_0)$ converges provided that the series $\sum |v_{n+1} - v_n|$ converges. But the convergence of the latter is obvious, since in view of (7) v_n is asymptotic with respect to n to an expression proceding in negative integral powers of n. Hence $\Delta(x_0)$ converges.

If we suppose that $\Delta(x_0)$ converges we may prove in a similar way that $\Omega(x_0)$ also converges.

Thus Theorem XI is established.

By means of the foregoing theorem we may readily determine the boundary lines of the regions of convergence and absolute convergence of the series $\Omega(x)$. For the discussion we make a separation into two cases according as $k \geqq m$ or $k < m$.

Suppose first that $k \geqq m$. From Theorems XI and IX it follows that the boundary line of the region of convergence [absolute convergence] of $\Omega(x)$ is the same as that of $\Delta(x)$ and hence the same as that of the series

$$\sum_{n=1}^{\infty} c_n e^{\sigma_n} e^{(\nu_{1n} \log n + \beta_{1n} + \gamma_{1n})x}.$$

From Theorem X it follows that the boundary of the region of convergence [absolute convergence] of the latter series, and hence of $\Omega(x)$, is the same as that of the series

$$\sum_{n=1}^{\infty} c_n e^{\sigma_n} e^{(km^{k-1}\log n)(\mu_k x)}. \tag{18}$$

In case $k < m$ (and hence $m \geqq 2$) it may be shown similarly that the boundary of the region of convergence [absolute convergence] of $\Omega(x)$ is the same as that of the series

$$\sum_{n=1}^{\infty} c_n e^{\sigma_n} e^{mn^{m-1}(\alpha_m x)}. \tag{19}$$

Let $\bar{\lambda}$ and $\bar{\mu}$ denote the abscissæ of convergence and absolute convergence respectively of the Dirichlet series

$$\sum_{n=1}^{\infty} b_n e^{-\lambda_n x}, \tag{20}$$

where $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ and λ_n approaches infinity as n increases indefinitely. Then for the values of $\bar{\lambda}$ and $\bar{\mu}$ we have*

$$\bar{\lambda} = \limsup_{\xi=\infty} \frac{\log |\sum_{\xi} b_\nu|}{\xi},$$

$$\mu = \limsup_{\xi=\infty} \frac{\log \sum_{\xi} |b_\nu|}{\xi},$$

where ξ denotes a continuously increasing variable and $\sum_{\xi} \beta_\nu$ stands for the sum of all β_ν whose suffix ν satisfies the relation

$$E(\xi) \leqq \lambda_\nu < \xi,$$

$E(\xi)$ denoting the greatest integer not greater than ξ. (If no suffix ν satisfies this relation then $\sum_{\xi} \beta_\nu$ has the value zero.)

Now (18) and (19) are special cases of (20) in which x is replaced by $-\mu_k x$ in the first instance and by $-\alpha_m x$ in the second instance. The boundary of the region of convergence [absolute convergence] of (20) is

$$R(x) = \bar{\lambda}\,[R(x) = \bar{\mu}].$$

As in the earlier part of this paper we denote the boundary of the region of

* These values for $\bar{\lambda}$ and $\bar{\mu}$ are due to T. Kojima, Tohoku Mathematical Journal, vol. 6 (1914): 134–139. If $\bar{\lambda}$ is not negative [not positive] we also have

$$\bar{\lambda} = \limsup_{n=\infty} \frac{\log\left|\sum_{\nu=1}^{n} b_\nu\right|}{\lambda_n} \qquad \left[\bar{\lambda} = \limsup_{n=\infty} \frac{\log\left|\sum_{\nu=n}^{\infty} b_\nu\right|}{\lambda_n}\right]$$

and corresponding formulæ for $\bar{\mu}$. The first of the above formulæ for $\bar{\lambda}$ is due to Cahen (Annales de l'École Normale Supérieure, ser. 3, vol. 11 (1894): 88–91) and the second to Schnee (Göttinger Nachrichten, 1910, pp. 1–42). They may be used to obtain formulæ alternative to those in Theorem XII, but less satisfactory on the whole.

convergence [absolute convergence] of $\Omega(x)$, and hence of (18) or (19), by $R(\sigma x) = \lambda\,[R(\sigma x) = \mu]$ where $\sigma = \mu_k$ or $\sigma = \alpha_m$ according as $k \geqq m$ or $k < m$. Hence by comparing (18) and (19) with (20) and making use of the relation (as to the boundary of the region of convergence [absolute convergence]) between $\Omega(x)$ on the one hand and (18) and (19) on the other hand, we see that the convergence [absolute convergence] number for $\Omega(x)$ is

$$(21)\quad \lambda = -\limsup_{\xi=\infty} \frac{\log\left|\sum_\xi c_\nu e^{\sigma_\nu}\right|}{\xi}\left[\mu = -\limsup_{\xi=\infty} \frac{\log \sum_\xi |c_\nu e^{\sigma_\nu}|}{\xi}\right]$$

where $\sum_\xi \beta_\nu$ stands for the sum of all β_ν whose suffix ν satisfies the condition

$$(22)\qquad E(\xi) \leqq k\nu^{k-1}\log\nu < \xi \quad \text{or} \quad E(\xi) \leqq m\nu^{m-1} < \xi,$$

according as $k \geqq m$ or $k < m$. (When no suffix ν satisfies the given condition $\sum_\xi \beta_\nu$ has the value 0.)

We shall now show that a suitable simplification of the foregoing values for λ and μ leads to the following theorem:

THEOREM XII. *Let $\lambda\,[\mu]$ denote the convergence [absolute convergence] number for the series $\Omega(x)$ or $\overline{\Omega}(x)$ so that the boundary of the region of convergence [absolute convergence] of this series is $R(\sigma x) = \lambda\,[R(\sigma x) = \mu]$, where σ is α_m or μ_k according as m is or is not greater than k. Then**

(*a*) *in case $k = 1$ and $m = 0$ or 1 we have*

$$\lambda = -\limsup_{\xi=\infty} \frac{\log\left|\sum_\xi c_\nu e^{\sigma_\nu}\right|}{\xi}, \qquad \mu = -\limsup_{\xi=\infty} \frac{\log \sum_\xi |c_\nu e^{\sigma_\nu}|}{\xi},$$

where $\sum_\xi \beta_\nu$ stands for the sum of all β_ν whose suffix ν satisfies the relation

$$e^{E(\xi)} \leqq \nu < e^{\xi};$$

(*b*) *in case $k > 1$ and $k \geqq m$ we have*

$$\lambda = \mu = -\limsup_{\nu=\infty} \frac{\log|c_\nu e^{\sigma_\nu}|}{k\nu^{k-1}\log\nu};$$

(*c*) *in case $k < m$ we have*

$$\lambda = \mu = -\limsup_{\nu=\infty} \frac{\log|c_\nu e^{\sigma_\nu}|}{m\nu^{m-1}}.$$

The conclusion in case (*a*) is obtained from (21) in an obvious manner.

In case (*b*) the relation between ν and ξ in $\sum_\xi \beta_\nu$ is determined by the first continued inequality in (22). For a given ξ sufficiently large it is clear that this relation is satisfied by not more than one value of ν. Hence λ and μ are equal, so that we may confine attention to λ alone. Moreover, it is clear that the first superior limit in (21) will be obtained if ξ is allowed to approach

* Note the alternative form of this theorem indicated in the corollary below. See also the preceding footnote.

infinity over those values alone for which a corresponding value of ν exists satisfying the first continued inequality in (22). But for such a value of ξ we may obviously replace ξ by $k\nu^{k-1}\log\nu$, where ν has the value corresponding to ξ in virtue of (22), without changing the value of the superior limit in consideration. Hence we are led to the conclusion stated in (*b*) of the theorem.

The conclusion in case (*c*) may be proved in the same way as that in case (*b*) was established.

From equations (7), (8), and (17) it follows that

$$g(n) \sim e^{\sigma_n} g_n(1), \tag{23}$$

the asymptotic relation being valid with respect to n increasing towards $+\infty$. Hence we are led at once to the following corollary of Theorem XII, at least so far as cases (*b*) and (*c*) are concerned:

COROLLARY. *In the conclusion of Theorem XII the quantity e^{σ_ν} (wherever it occurs) may be replaced by $g(\nu)$ without destroying the truth of the statements made.*

An almost immediate proof of the theorem of this corollary may also be obtained by comparing the series

$$\sum_{n=0}^{\infty} c_n\, g(n)\, e^{-\sigma_n}\, g(x+n)$$

with the series $\Omega(x)$. By means of Lemma I and relation (23) one proves readily that the two series have the same non-exceptional points of convergence.

Let us consider the instance of the series $\overline{\Omega}(x)$ in which $g(x) = e^{x^2}$. Then $\sigma = 1$ and $\sigma_n = n^2$. If we replace $c_n e^{n^2}$ by a_n we may write this instance of $\overline{\Omega}(x)$ in the form

$$\sum_{n=0}^{\infty} a_n e^{2x\cdot n}.$$

Replacing e^{2x} by z we have an ordinary ascending power series in z. Applying conclusion (*c*) of Theorem XII we find the Cauchy-Hadamard expression

$$\limsup_{\nu=\infty} \sqrt[\nu]{|a_\nu|}$$

for the reciprocal of the radius of convergence of the power series in z, so that our theorem is a generalization of the Cauchy-Hadamard theorem for the radius of convergence of a power series.

Let us consider a second special instance of the series $\overline{\Omega}(x)$, namely, that obtained by putting $g(x) = 1/\Gamma(x)$. If we write $(n-1)!\,a_n$ for c_n the series takes essentially the form

$$\sum_{n=1}^{\infty} \frac{(n-1)!\, a_n}{x(x+1)(x+2)\cdots(x+n-1)}.$$

From case (a) of Theorem XII, as modified by the corollary, we see that the abscissa λ [μ] of convergence [absolute convergence] is given by the relation

$$\lambda = -\limsup_{\xi=\infty} \frac{\log|\sum_{\xi} a_\nu|}{\xi} \left[\mu = -\limsup_{\xi=\infty} \frac{\log \sum_{\xi}|a_\nu|}{\xi}\right],$$

ν and ξ being related as in the theorem.

The strip between the boundary line of convergence and the boundary line of absolute convergence of the Dirichlet series (20) is known* to have a width at most equal to

$$\limsup_{\nu=\infty} \frac{\log \nu}{\lambda_\nu}.$$

Applying this fact to series (18) and (19) and remembering their relation to the series $\Omega(x)$ we have the following theorem:

THEOREM XIII. *The boundary lines of the regions of convergence and absolute convergence of the series* $\Omega(x)$ [$\bar{\Omega}(x)$] *coincide unless* $k = 1$ *and* $m = 0$ *or* 1; *and in the latter case the width of the region of conditional convergence is at most* $1/R(\mu_k)$.

From the theory of Dirichlet series and the relation between $\Omega(x)$ and series (18) it follows in the latter case that this strip of conditional convergence may be non-existent or may have its maximum width or may have an intermediate width, examples of Dirichlet series being readily constructed corresponding to each possibility.

7. FURTHER RELATIONS BETWEEN THE SERIES $\Omega(x)$ AND $\Delta(x)$ WHEN $k = 1$ AND $m = 0$ OR 1

A particularly important subclass of the series $\Omega(x)$ and $\bar{\Omega}(x)$ is obtained through restricting $g(x)$ by the condition that k shall be unity and m either zero or unity; for this subclass contains the important series to which solutions of linear homogeneous difference equations give rise. For this special subclass the relation between the series $\Omega(x)$ and $\Delta(x)$ is much deeper than is indicated by the theorems of § 6, as we shall now show. For this case we shall denote the series by $\omega(x)$ and $\delta(x)$, so that we have†

$$\omega(x) = \sum_{n=1}^{\infty} c_n\, g(x+n), \qquad \delta(x) = \sum_{n=1}^{\infty} c_n\, u(x,n), \tag{24}$$

where

* Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, p. 734.

† Here we take $c_0 = 0$ as a matter of convenience.

$$u(x, n) = n^{\mu_0+\mu_1(x+n)}\, e^{\alpha_0+\mu_1 x+\beta(x+n)},$$

β having the value α_1 or 0 according as $m = 1$ or $m = 0$. For $g(x+n)$ we have with respect to n the asymptotic relation

$$g(x+n) \sim u(x, n)\left(1 + \frac{h_1(x)}{n} + \frac{h_2(x)}{n^2} + \cdots\right),$$

the functions $h_1(x)$, $h_2(x)$, $\cdots$ being polynomials in x which are independent of n.

Let us write

$$\frac{g(x+n)}{u(x, n)} = 1 + \frac{h_1(x)}{n} + \cdots + \frac{h_\tau(x)}{n^\tau} + \frac{M_\tau(x, n)}{n^{\tau+1}},$$

where τ is a positive integer. Then $M_\tau(x, n)$ approaches a finite value when x is held fixed and n approaches infinity. Thence we have

$$\omega(x) - \delta(x) - h_1(x)\delta_1(x) - \cdots - h_\tau(x)\delta_\tau(x) = \sum_{n=1}^{\infty} \frac{c_n}{n^{\tau+1}} M_\tau(x, n)\, u(x, n), \tag{25}$$

where

$$\delta_i(x) = \sum_{n=1}^{\infty} c_n n^{-i} u(x, n) = e^{-i-(\beta i/\mu_1)}\, \delta(x - i/\mu_1) \quad (i = 1, 2, \cdots, \tau).$$

Now, let λ denote the common convergence number of $\omega(x)$ and $\delta(x)$. Then $\lambda + i$ is the convergence number for the series $\delta_i(x)$. Furthermore, it is easy to see also through aid of Theorem XIII that the absolute convergence number, and hence the convergence number, of the series in the second member of (25) is not less than $\lambda + \tau$.

With these results in hand let us consider further the character of the functions $\omega(x)$ and $\delta(x)$ which are defined in an appropriate half-plane by the series $\omega(x)$ and $\delta(x)$ respectively. Let x_0 be any point such that $\delta(x)$ is analytic at the points

$$x_0 - \frac{1}{\mu_1}, \quad x_0 - \frac{2}{\mu_1}, \quad x_0 - \frac{3}{\mu_1}, \quad \cdots. \tag{26}$$

Then the functions $\delta_i(x)$, $i = 1, 2, 3, \cdots$, are analytic at the point x_0. Moreover, only a finite number of the points (26) fail to lie within the common region of convergence of the series $\omega(x)$ and $\delta(x)$. From these considerations and relation (25) it follows readily that the functions $\omega(x)$ and $\delta(x)$ are either both regular or both singular at the point x_0, provided that x_0 is non-exceptional for the series $\omega(x)$.

The foregoing results may be gathered into the following theorem:

THEOREM XIV. *Let $\omega(x)$ and $\delta(x)$ be functions which are defined in an appropriate half-plane by the series in* (24). *If $\delta(x)$ is analytic at each of the points* (26) *and no one of these points is exceptional for the series $\omega(x)$, then the functions $\omega(x)$ and $\delta(x)$ are both regular or both singular at the point $x = x_0$.*

By means of the last theorem we may readily prove the following:

THEOREM XV. *If the numbers*

$$c_n n^{\mu_0+\mu_1 n} e^{\beta_n},$$

with the exception at most of a finite number of them, are real and non-negative and if the convergence number λ of the series $\omega(x)$ is finite, then the function $\omega(x)$ defined by this series has a singularity at the point $x = \lambda/\mu_1$ on the boundary line of convergence, provided that this point is non-exceptional for the series $\omega(x)$.

The truth of this theorem is an immediate consequence of Theorem XIV and the well-known fact (see Landau's Handbuch, l. c., p. 880) that the function $\delta(x)$ defined by the Dirichlet series in (24) has a singularity at $x = \lambda/\mu_1$.

From the corresponding property of the Dirichlet series $\delta(x)$ it follows that the boundary line of convergence of the series $\omega(x)$ does not always pass through a singular point of the function $\omega(x)$ defined by the series.

8. ON THE UNIQUENESS OF EXPANSIONS IN SERIES $\Omega(x)$ AND $\overline{\Omega}(x)$

In order that series of a given type shall afford a convenient tool for investigating the properties of functions representable in terms of them it is necessary that a given function shall have not more than one expansion in series of a specified form in that type, the form being sufficiently restricted. That the series $\Omega(x)$ and $\overline{\Omega}(x)$, for a given function $g(x)$, do not always possess this desirable property is shown by the special case in which

$$g(x) = \Gamma(x).$$

If in the series $\overline{\Omega}(x)$, for this special case, we replace x by $-z$ the series takes the form

$$\gamma_0 + \sum_{n=1}^{\infty} \gamma_n z(z-1)\cdots(z-n).$$

Pincherle* has shown that a series of this form may represent 0 without having all its coefficients equal to zero and that when a function admits an expansion in a series of such form it admits an infinity of such expansions.

The object of this section is to exhibit an extensive subclass of each of the series $\Omega(x)$ and $\overline{\Omega}(x)$ in which the uniqueness expansion theorem is valid

*Atti della Reale Accademia dei Lincei, Rendiconti, ser. 5, vol. 11, pp. 417–426.

in such wise that no function has more than one expansion in a series $\Omega(x)$ or a series $\overline{\Omega}(x)$ for a given function $g(x)$ of the type through which these subclasses arise.

As before we employ σ to denote α_m or μ_k according as m is or is not greater than k. We have seen that the points in the interior of the region of convergence of a series $\Omega(x)$ or a series $\overline{\Omega}(x)$ are all the points x satisfying the condition $R(\sigma x) < \lambda$ where λ is an appropriately determined constant. If the real part of σ is negative and x is any fixed number then it is clear that only a finite number (at most) of the points $x, x+1, x+2, \cdots$ fail to lie within the region of convergence of the series; but if $R(\sigma)$ is positive an infinite number of these points lie outside of the region ot convergence. Moreover, if $R(\sigma)$ is negative it is clear that $g(x+n)$ approaches zero as n increases indefinitely. These considerations taken in connection with the form of $\Omega(x)$ and $\overline{\Omega}(x)$ direct attention to the case in which $R(\sigma)$ is negative. We shall now show that the desired uniqueness theorem is true in case this condition is satisfied.

As a preliminary to this end let us consider the positive real-valued function $h(u)$ of the real variable u defined by the relation

$$h(u) = u^{p(u)} e^{q(u)},$$

where

$$p(u) = \nu_0 + \nu_1 u + \nu_2 u^2 + \cdots + \nu_k u^k,$$

$$q(u) = \beta_0 + \beta_1 u + \beta_2 u^2 + \cdots + \beta_m u^m,$$

the constants ν and β having the values

$$\nu_j = R(\mu_j), \quad j = 0, 1, \cdots, k; \qquad \beta_j = R(\alpha_j), \quad j = 0, 1, \cdots, m.$$

Since $R(\sigma)$ is negative we have ν_k negative or β_m negative according as $k \geqq m$ or $k < m$. Thence it is easy to see that *the ratio* $h(u+1)/h(u)$ *constantly decreases as* u *increases*, provided at least that u is sufficiently large.

Now, let us suppose that a given function $f(x)$ has the expansions

$$f(x) = \sum_{n=0}^{\infty} c_n g(x+n) = \sum_{n=0}^{\infty} d_n g(x+n),$$

both expansions being valid for $R(\sigma x)$ less than some number λ_1. We shall show that $c_n = d_n$ for every value of n. For this purpose it is obviously sufficient to show that if the relation

$$\sum_{n=0}^{\infty} \gamma_n g(x+n) = 0 \tag{28}$$

is valid for every non-exceptional value of x for which $R(\sigma x) < \lambda_1$ then $\gamma_0 = 0 = \gamma_1 = \gamma_2 = \cdots$. From equation (28) and the asymptotic character of the function $g(x)$ and the relation between $g(x)$ and $h(u)$ we see that

$$|\gamma_0| \leqq \sum_{n=0}^{\infty} |\gamma_n| \left| \frac{g(u+n)}{g(u)} \right| < M \frac{h(u+1)}{h(u)} \sum_{n=1}^{\infty} |\gamma_n| \frac{h(u+n)}{h(u+1)},$$

M being an appropriately determined constant. That the series in the last member of this relation converges when u is sufficiently large follows at once from the fact that its absolute convergence number is the same as that of the series $\Delta(x)$ associated with the series $\Omega(x)$ in equation (28). We shall show that the sum of this series is bounded as u increases. It may be written in the form

$$\sum_{n=1}^{\infty} |\gamma_n| \frac{h(u+2)}{h(u+1)} \cdot \frac{h(u+3)}{h(u+2)} \cdot \cdots \cdot \frac{h(u+n)}{h(u+n-1)}.$$

Now, when u is sufficiently large each of the fractions $h(u+i+1)/h(u+i)$ decreases as u increases; and hence the sum of this series does not increase indefinitely as u increases. Therefore from (28) we have a relation of the form

$$|\gamma_0| < \overline{M} \frac{h(u+1)}{h(u)},$$

where $\overline{M}$ is a constant. Letting u increase indefinitely we see from this that $\gamma_0 = 0$. If in (28) we replace γ_0 by 0 and x by $x - 1$ we may proceed as before to prove that $\gamma_1 = 0$. Likewise we have in order

$$\gamma_2 = 0 = \gamma_3 = \gamma_4 = \cdots.$$

Thus we are led to the following theorem:

THEOREM XVI. *Let σ denote α_m or μ_k according as m is or is not greater than k. Let $g(x)$ be a function for which $R(\sigma)$ is negative. Then a given function $f(x)$ has not more than one expansion in a series $\Omega(x)$ or a series $\overline{\Omega}(x)$ for this given function $g(x)$.*

In case a given function $f(x)$ has an expansion in the form

$$f(x) = \sum_{n=0}^{\infty} c_n g(x+n)$$

and $R(\sigma)$ is negative, the coefficients $c_0, c_1, \cdots$ may readily be determined by means of the properties of the function $f(x)$ itself. In fact, in view of the argument by means of which the foregoing theorem was proved it is easy to see that we have

$$c_0 = \lim_{n=+\infty} \frac{f(n)}{g(n)},$$

$$c_1 = \lim_{n=+\infty} \frac{f(n) - c_0 g(n)}{g(n+1)},$$

$$c_2 = \lim_{n=+\infty} \frac{f(n) - c_0 g(n) - c_1 g(n+1)}{g(n+2)},$$

. . . . . .

In case a given function $f(x)$ has an expansion in the form of a series $\overline{\Omega}(x)$ and $R(\sigma)$ is negative, the coefficients in the expansion may obviously be determined in a similar manner.

UNIVERSITY OF ILLINOIS,
October, 1915

THE GEOMETRIES ASSOCIATED WITH A CERTAIN SYSTEM OF CREMONA GROUPS*

BY

JOHN WESLEY YOUNG AND FRANK MILLETT MORGAN

Introduction. A geometry is defined by a class of elements S and a group of transformations G operating in a one-to-one reciprocal way on the elements of S and transforming S into itself. The geometry (S, G) then consists of such properties of figures (i. e., combinations of elements of S) as remain invariant under the transformations of G. This is in brief Klein's classical principle.†

The most familiar geometries having the point as the primary element of space which have received systematic study are:

(1) The geometries defined by the general *projective groups* in one, two, . . . , n dimensions, and their subgroups.

(2) The geometries defined by the *inversion groups* in two, three, and to some extent n dimensions.

The object of the present paper is to call attention to a system of groups $G_2, G_3, \cdots, G_n$ of Cremona transformations in point spaces $S_2, S_3, \cdots, S_n$ of two, three, . . . , n dimensions and to develop elementary properties of the corresponding geometries. This system of groups includes for the case $n = 2$ the group G_2 of direct circular transformations in the plane; the corresponding plane geometry is then closely related to the ordinary inversion geometry in the plane. The associated geometries in three or more dimensions form, therefore, *a direct extension of the inversion geometry in a plane*, different from the inversion geometries in the spaces of three or more dimensions. The elementary curves in the geometry (S_2, G_2) are the conics through two fixed points in a projective plane (in the ordinary orientation of the inversion geometry these points are the circular points at infinity so that the conics referred to are circles). In the geometries (S_3, G_3) the elementary

* Presented to the Society, under different titles, by J. W. Young, December 28, 1911, and by J. W. Young and F. M. Morgan, April 26, 1913, and April 24, 1915.

† Felix Klein, *Vergleichende Betrachtungen über neuere geometrische Forschungen*, Erlangen, 1873. Reprinted Mathematische Annalen, Vol. 43 (1892), pp. 63–100. English translation by M. W. Haskell, Bulletin of the American Mathematical Society, Vol. 2 (1893), pp. 215–49.

curves are twisted cubics; in the geometries (S_n, G_n) they are normal curves of order n in S_n.

Definition of groups $G_2, G_3, \cdots, G_n$. It will be sufficient to define in detail the groups G_2 and G_3. The extension to G_n will then be sufficiently obvious.

The Group G_2. Let U_1, U_2 be the distinct centers of two pencils of lines in the (real or complex) projective plane Σ_2. Every point P in the plane not on the line $U_1 U_2$ determines and is determined by a unique line in each of the pencils U_1, U_2. The point P is then determined by the two lines l_1 (through U_1) and l_2 (through U_2) and we may write $P = (l_1, l_2)$. Let π_1 be any projectivity in the pencil U_1 and π_2 any projectivity in the pencil U_2 and let $\pi_1(l_1) = l_1'$ and $\pi_2(l_2) = l_2'$. The transformation $\pi = (\pi_1, \pi_2)$ is then defined as the transformation which transforms $P = (l_1, l_2)$ into $P' = (l_1', l_2')$. π is a transformation of G_2, and G_2 consists of all such transformations π which can be obtained by varying the *component projectivities* π_1, π_2. G_2 is a group of (in general) quadratic transformations which transforms into itself transitively the system of all conics through U_1 and U_2.

The Group G_3. Let u_1, u_2, u_3 be the nonconcurrent axes of three pencils of planes in projective space Σ_3. Every point P in Σ_3 (not on the quadric surface determined by u_1, u_2, u_3) determines a unique plane in each of the pencils u_1, u_2, u_3. The point P is then determined by the three planes α_1 (through u_1), α_2 (through u_2), α_3 (through u_3) and we may write $P = (\alpha_1, \alpha_2, \alpha_3)$. Let π_1 be a projectivity in the pencil u_1, π_2 a projectivity in the pencil u_2, and π_3 a projectivity in the pencil u_3; and let $\pi_i(\alpha_i) = \alpha_i'$ $(i = 1, 2, 3)$. The transformation $\pi = (\pi_1, \pi_2, \pi_3)$ is then defined as the transformation which transforms $P = (\alpha_1, \alpha_2, \alpha_3)$ into $P' = (\alpha_1', \alpha_2', \alpha_3')$. π is in general a cubic transformation and is a transformation of G_3. G_3 consists of all such transformations π which can be obtained by varying the *component projectivities* π_1, π_2, π_3. G_3 transforms into itself transitively the system of all twisted cubics having u_1, u_2, u_3 as bisecants.

The group $G_n (n > 3)$ is defined similarly in Σ_n by means of projectivities in n pencils of Σ_{n-1}'s. They are groups of Cremona transformations of order n (in general), which transform transitively into themselves certain systems of normal curves of order n in Σ_n.

It should be noted that for $n > 2$, G_n has several distinct forms. Thus G_3 has four distinct forms according to the figure formed by the lines u_1, u_2, u_3. These lines may be skew to each other (G_3^1), or two only may intersect (G_3^2), or one of the lines may intersect each of the other two without their being coplanar (G_3^3), or finally u_1, u_2, u_3 may form a triangle (G_3^4). The cubic transformations thus defined have been extensively studied by Noether and

others.* The groups of such transformations have, however, received but little attention hitherto.†

In what follows explicit reference is generally made only to the case $n = 3$. Most of our theorems, however, apply either without any change or with very obvious changes to all cases, $n \geqq 2$. The following notational scheme enables us readily to assign the proper generality to the later results. For the case $n = 2$ there is only one group G_2 while for the case $n = 3$ there are four groups $G_3^1, G_3^2, G_3^3, G_3^4$: the symbol G_3 is used to represent any one of these last four groups. Similarly G_n represents any one of the groups possible for a general n; while, when the number of dimensions is not significant, we shall simply use G to represent any one of the whole system of groups.

The space S. The group G_3 has been defined in terms of, and as acting on, the (real or complex) projective point space Σ_3 of three dimensions. It does not, however, operate on the points of Σ_3 in a one-to-one reciprocal way. Our first problem is, then, to define a space S_3, on the primary elements of which G_3 operates in a one-to-one reciprocal way. The definition of such a space is very simple, for we need only keep in mind the fact that G_3 does operate in a reciprocally one-to-one way on the triples of planes $\alpha_1, \alpha_2, \alpha_3$. We may therefore simply define these *plane-triples* as the "points" of our space S_3.‡ However for the sake of greater intuitional satisfaction, we prefer to replace these plane triples by "points," proper and improper, determined by them in a one-to-one reciprocal way.

We have first as *proper points* of S_3, all the points of Σ_3 which are not on the quadric surface $(u_1 u_2 u_3)$.§

We have secondly, *improper points*, in general as follows:

(1) *Straight lines* when the planes of the plane-triple $(\alpha_1, \alpha_2, \alpha_3)$ intersect in a line. These lines are the lines meeting u_1, u_2, and u_3.

(2) *Point-planes*, consisting of a point on u_i $(i = 1, 2, 3)$ with a plane through u_i. These occur when two of the planes $(\alpha_1, \alpha_2, \alpha_3)$ meet on u_i.

---

* Noether, *Über die eindeutigen Raumtransformationen, insbesondere in ihrer Anwendung auf die Abbildung algebraischer Flächen*, Mathematische Annalen, Bd. 3 (1871), S. 547–580.

Doehlemann, *Über eine einfache eindeutige Raumtransformation dritter Ordnung*, Münchener Berichte, Bd. 24 (1894), S. 41–50.

Doehlemann, *Geometrische Transformationen*, Teil (2), S. 308.

Sturm, *Die Lehre von den Geometrischen Verwandtschaften*, IV, § 127.

† Fano, *I gruppi di Jonquières generalizzati*, Torino Memorie (2), vol. 48 (1898), p. 221.

Hudson, *The* (3, 3) *Transformations in Space*, Proceedings of the London Mathematical Society, 1913.

‡ For G_n we should define the "points" of S_n as the corresponding n-ples of Σ_{n-1}'s.

§ This quadric degenerates in the cases G_3^2, G_3^3 into two planes; in case of G_3^4 into a single plane.

When two of the lines u_1, u_2, u_3 intersect, other forms of improper points arise as will be seen by a direct consideration of the cases in question.

To define completely the spaces S_3^1, S_3^2, S_3^3, S_3^4 associated respectively with G_3^1, G_3^2, G_3^3, G_3^4, it is only necessary to list the improper points associated in each case with the proper points to form the space S_3 in question. We give below such a definition for S_3^1 because of its simplicity, and for S_3^4 as it is in some respects the most interesting.

The space S_3^1 of G_3^1. (The lines u_1, u_2, u_3 are skew to each other.)

(1) *Proper points:* All the points of the projective Σ_3 not on the quadric $(u_1 u_2 u_3)$.

(2) *Improper points:*

(i) All the *lines* meeting u_1, u_2, u_3.

(ii) All the *point-planes* consisting of a point P on u_i $(i = 1, 2, 3)$ together with any plane through u_i distinct from the tangent plane to the quadric $(u_1 u_2 u_3)$ at P.

If the projective Σ_3 is real, there are ∞^1 improper points of the *first kind*, and three systems of ∞^2 improper points of the *second kind*.

The space S_3^4 of G_3^4. (The lines u_1, u_2, u_3 form a triangle.)

(1) *Proper points:* All points of the projective Σ_3 not on the plane of the triangle.

(2) *Improper points:*

(i) All *linear elements* consisting of a vertex U_i $(i = 1, 2, 3)$ of the triangle $u_1 u_2 u_3$ and a line through U_i, not in the plane of u_1, u_2, u_3.

(ii) All *line-planes* consisting of a line u_i $(i = 1, 2, 3)$ and any plane through u_i distinct from the plane of $u_1 u_2 u_3$.

(iii) The *plane* of the triangle $u_1 u_2 u_3$.

If S_3 is real, there are three systems of ∞^2 improper points of the *first kind*, three systems of ∞^1 improper points of the *second kind*, and one improper point of the *third kind*.

Throughout the remainder of the paper unless otherwise specified the word "point" will refer to an element, proper or improper, of the space S associated with the group G.

Analytic representation of G. The associated hyper-complex number system. If we establish a projective *scale* in each of the pencils u_i $(i = 1, 2, 3)$ by assigning the numbers $0, 1, \infty$ to any three distinct planes through u_i,* then every point P in S_3 may be denoted by three coördinates (x_1, x_2, x_3), x_i being the coördinate of the plane through P in u_i in the scale just referred to. The transformations of the group G_3 are then given by the relations:

$$x_i' = \frac{a_i x_i + b_i}{c_i x_i + d_i}, \qquad a_i d_i - b_i c_i \neq 0 \qquad (i = 1, 2, 3). \tag{1}$$

* Veblen and Young, *Projective Geometry*, Vol. 1, Boston, 1910, p. 141.

If $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ are any two points of S_3 (neither of which is on a plane ∞) a hypercomplex algebra N_3* is defined by the relations,

$$A + B = (a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3),$$
$$A \cdot B = (a_1 b_1,\ a_2 b_2,\ a_3 b_3). \tag{2}$$

It is at once evident from this definition that N_3 satisfies the ordinary associative, commutative, and distributive laws as to addition and multiplication.† The nil-factors (divisors of zero) are the elements (x_1, x_2, x_3) in which one or more of the x_i $(i = 1, 2, 3)$ are zero. Moreover it should be noted that the elements of the form (x, x, x) form a sub-algebra R_3 of N_3 isomorphic with the algebra of ordinary real or complex numbers according as x ranges over the system of real or complex numbers. The numbers of R_3 may be conveniently designated as the *scalars* of N_3.

In the terms of this algebra, the transformations of G_3 are given by the relations:

$$X' = \frac{AX + B}{CX + D} \qquad (AD - BC \neq \text{a nil factor}), \tag{3}$$

where A, B, C, D represent any elements of N_3.

The extension of these results to any group G_n is obvious.‡

Invariant of G. Cross-ratio. Associated with every ordered set of four points

$$A = (\alpha_1, \alpha_2, \alpha_3), \qquad B = (\beta_1, \beta_2, \beta_3),$$
$$C = (\gamma_1, \gamma_2, \gamma_3), \qquad D = (\delta_1, \delta_2, \delta_3)$$

are three cross-ratios $\rho_i = \mathrm{R}_i(\alpha_i \beta_i, \gamma_i \delta_i)$ $(i = 1, 2, 3)$. The *cross ratio symbol* $\rho = (\rho_1, \rho_2, \rho_3) = \mathrm{R}_\rho(AB, CD)$ is then invariant under G_3, and is hence fundamental in the geometry (S_3, G_3). It will be noted that the cross-ratio symbol ρ is an element of the hypercomplex algebra N_3. This close connection of (S, G)§ with the projective geometry in a one-dimensional

* In papers by J. W. Young, *A New Formulation for General Algebra*, read before the American Mathematical Society, Dec. 28, 1911, and Apr. 26, 1913 (abstracts in Bulletin, vol. 18, Feb., 1912; vol. 19, July, 1913), the groups here discussed were discovered as a natural result of the investigations of these papers.

† The algebra N_3 is not invariant under G, that is to say if $A + B = C$, and if an operation of the group sends A, B, C into A', B', C' then, in general, $A' + B' \neq C'$.

‡ The representation of G in the form (3) exhibits strikingly the analogy of G with the group G_2 of direct circular transformations in a plane. The corresponding algebra N_2 is equivalent to the algebra of ordinary complex numbers if properly orientated in the *real* S_2. The representation of the real circular transformations in a plane by linear fractional transformations on a complex variable is well known.

§ Reference may be made to a paper by J. W. Young, *The Geometry of Chains on a Complex Line*, Annals of Mathematics, series 2, vol. 11, pp. 33–48, 1909–10. The methods and the results of this paper and the present one bear a close analogy. Indeed, the geometries (S, G) are in a sense geometries on a hypercomplex line.

form at once suggests a method of approach to the geometry (S, G). Certain conceptions and the associated terminology of one-dimensional projective geometry may at once be carried over. For example, two pairs of points AB, CD of S_3 form a harmonic set $H(AB, CD)$ if the sets of planes $(\alpha_i \beta_i, \gamma_i \delta_i)$ $(i = 1, 2, 3)$ are harmonic: i. e., if $\mathrm{R}_{\gamma}(AB, CD) = (-1, -1, -1)$. It is natural to designate a scalar (x, x, x) of N_3 by the number x. With this notation we may say that AB, CD form a harmonic set if

$$\mathrm{R}_{\gamma}(AB, CD) = -1.$$

The fundamental theorem of G. Let us call two points $A = (\alpha_1, \alpha_2, \alpha_3)$ and $A' = (\alpha_1', \alpha_2', \alpha_3')$ of S_3 *co-axial* if they determine the same plane in any one of the pencils u_i, i. e., if either $\alpha_1 = \alpha_1'$, or $\alpha_2 = \alpha_2'$, or $\alpha_3 = \alpha_3'$.*

THEOREM 1. *If A, B, C and A', B', C' are any two sets of three points of S, such that no two points of either set are co-axial, then there exists one and only one transformation π of G, such that $\pi(ABC) = A'B'C'$.*

This theorem is a direct consequence of the definition of G and the fundamental theorem of projective geometry.

The simple curves of (S, G). Let A, B, C be any three points of S_3 no two of which are co-axial, and consider the locus of a point P such that $\mathrm{R}_{\gamma}(AB, CP)$ is a scalar $\lambda = (\lambda, \lambda, \lambda)$ of N_3, where λ is any real or complex variable. This locus may also be defined as the locus of points determined by the triples of homologous planes in the projectivities

$$\alpha_1 \beta_1 \gamma_1 \barwedge \alpha_2 \beta_2 \gamma_2 \barwedge \alpha_3 \beta_3 \gamma_3.$$

A curve thus defined we will call a *C-curve*. In view of the following theorems, which are immediate consequences of the definition and Theorem 1, the C-curves are *simple one dimensional elements* of the geometry (S, G).

THEOREM 2. *If A, B, C are any three points of S, no two of which are co-axial, there exists one and only one C-curve containing A, B, and C.*

The C-curve through A, B, C will be denoted by $|ABC|$.

THEOREM 3. *Any two C-curves are equivalent under the group G.*

The C-curves in S_3 are, in general, twisted cubics (in Σ_3) having the axes u_1, u_2, u_3 as *bisecants* and any such twisted cubic is a C-curve.† However

* In S_n, $A = (\alpha_1, \alpha_2, \alpha_3, \cdots \alpha_n)$ and $A' = (\alpha_1', \alpha_2', \alpha_3', \cdots \alpha_n')$ are co-axial if one of the relations $\alpha_i = \alpha_i'$ holds.

† That the locus of the point of intersection of the homologous planes of three projective axial pencils is in general a twisted cubic having the axes of the pencils as bisecants, seems to have been first proved by Chasles, Comptes Rendus, vol. 45 (1857), p. 189 and Journal de Mathématiques, 2d ser., vol. 2 (1857), p. 397. See also Reye, *Geometrie der Lage*, vol. 2, pp. 91–93; Sturm, *Geometrische Verwandtschaften*, vol. I, p. 306. In S_n the C-curves are normal curves of order n in Σ_n, having as $(n-1)$-secants the "axes" of the generating pencils of Σ_{n-1}'s, and every such normal curve is a C-curve: see Bertini, *Introd. alla geometria proiettiva degli iperspazi*, p. 275.

these cubics degenerate under certain conditions into conics and straight lines. It is quite unnecessary for the further development of this paper, to describe these curves in detail; for, all later theorems are consequences merely of the definitions and Theorems 1, 2, 3. It seems desirable nevertheless to list the C-curves completely, at least for S_3^1 and S_3^4.

The class of C-curves in S_3^1 consist of: (1) All the *twisted cubics* of Σ_3 having u_1, u_2, u_3 as bisecants. These are characterized in S_3^1 by the fact that none of them contains an improper point of the first kind. (2) All the *conics* of Σ_3 not on the quadric $u_1 u_2 u_3$ but meeting u_1, u_2, and u_3 each once. These are characterized in S_3^1 by the fact that each such C-curve contains one and only one improper point of the first kind. (3) All the *straight lines* of Σ_3 not on the quadric $u_1 u_2 u_3$ nor meeting u_1, u_2, u_3. These are characterized by the fact that each such C-curve contains two and only two distinct improper points of the first kind. (4) The *locus of all improper points of the first kind.*

The class of C-curves for S_3^4 consists of: (1) All *twisted cubics* of Σ_3 containing the vertices U_1, U_2, U_3 of the triangle u_1, u_2, u_3. These are characterized in S_3^4 by the fact that every such C-curve contains three improper points of the first kind and no other improper points. (2) All *conics* of Σ_3 not on the plane u_1, u_2, u_3 but passing through U_i $(i = 1, 2, 3)$ and meeting the opposite side of the triangle. These are characterized in S_3^4 by the fact that each such conic contains an improper point of the first kind, one of the second kind and no other improper points. (3) All *straight lines* of S_3 not through a side of the triangle $u_1 u_2 u_3$. These are characterized in S_3 by the fact that each such C-curve contains the improper point of the third kind and no other improper point.*

A fundamental property of a C-curve, which follows at once from the definition, is given by

THEOREM 4. *If A, B, C are three distinct points of a C-curve, the harmonic conjugate of A with respect to B and C is on the curve.*

Properties of G. The projective structure of the group G makes it possible to infer numerous properties of the group almost immediately. The enumeration of subgroups and their properties, questions of the conjugacy of trans-

* *Metric orientation.* In the usual orientation of the real geometry (S_2, G_2) the two points $U_1 U_2$ coincide with the circular points at infinity in the plane Σ_2. The system of C-curves (the conics through U_1, U_2) then appear as the system of circles (and straight lines) in S_2. An analogous orientation of the geometry (S_3^4, G_3^4) in real Σ_3 may be obtained by letting the triangle u_1, u_2, u_3 consist of two conjugate tangents to the absolute conic at infinity and their real chord of contact. If U_1 be the real intersection of these tangents, and if we choose the direction determined by U_1 to be vertical, the system of real C-curves consists of (1) The twisted cubics on all right circular cylinders whose axes are vertical. (2) The rectangular hyperbolas one of whose asymptotes is vertical. (3) The straight lines which are neither vertical nor horizontal.

formations, etc., while simple, we pass over as not of immediate concern in the present paper. The following results are needed, however, in what follows.

THEOREM 5. *Every transformation of G transforms a harmonic set of points into a harmonic set of points.*

THEOREM 6. *Two transformations of G with the same double points are commutative.*

The involutions, I, of G (i. e., transformations I such that $I^2 = 1$, $I \neq 1$) are of special interest. Since every component involution in the generating pencils has two distinct double planes in (complex) Σ_3, it is clear that every involution I of G_3 has in general $2^3 = 8$ double points in complex S_3. These eight double points consist of four pairs, the two points of each pair not being co-axial. Such a pair of non-co-axial double points will be called a pair of *conjugate double points** of I and will be denoted by $M_i N_i$ $(i = 1, 2, 3, 4)$.† We have then,

THEOREM 7. *In an involution I of G any point A and its image A′ are harmonic with respect to any pair of conjugate double points $M_i N_i$ of I. Symbolically $I(A) = A' = H(A, M_i N_i)$.*

THEOREM 8. *An involution I of G is completely determined by two pairs of homologous points; and, in particular, by a pair of conjugate double points.*

THEOREM 9. *Any transformation π of G is the product of two involutions of G.*

THEOREM 10. *If A, A′ is a pair of homologous points of an involution I, the involution having A, A′ for double points is commutative with I.*

COROLLARY. *The product of two involutions with conjugate double points A, A′ and B, B′, respectively, transforms into itself the involution in which A, A′ and B, B′ are homologous pairs.*

Invariant Curves. It will be supposed in what follows that S is complex.

THEOREM 11. *If a C-curve through $M_i N_i$ and one through $M_j N_j$ $(i \neq j)$ [$M_i N_i$ and $M_j N_j$ being conjugate double points of an involution I] have one point in common they have a second point in common.*

Let C and C' be the C-curves and let P be their common point.

Since the harmonic conjugate of P is on both C and C' the curves have a second point in common.

THEOREM 12. *Every C-curve containing a pair of conjugate double points $M_i N_i$ of an involution I is invariant under I.*

Let C be any C-curve containing the double points $M_i N_i$ and let A be any

* An involution I of G_n has in general 2^n double points in (complex) S_n consisting of 2^{n-1} pairs of non-co-axial points, i. e., *conjugate double points.*

† Intuitionally, the situation is clarified by thinking of the eight double points as the eight corners of a box. Conjugate double points are diagonally opposite corners.

point of $C = |M_i N_i A|$. Then $I(A) = A' = H(A, M_i N_i)$. Hence A' is a point of C.

THEOREM 13. *Every C-curve invariant under an involution I passes through a pair of conjugate double points of I.*

COROLLARY. *Through every point in S_3 distinct from $M_i N_i$ $(i = 1, 2, 3, 4)$ pass four and only four C-curves invariant under I.*

THEOREM 14. *If a C-curve C not passing through a given pair of conjugate points $M_i N_i$ of an involution I meets two C-curves through $M_i N_i$ in two pairs of points harmonic with regard to $M_i N_i$ then C passes through a pair of double points $M_j N_j$ $(i \neq j)$.*

Let A, A' and B, B' be the two pairs of points. Then $H(M_i N_i, AA')$ and $H(M_i N_i, BB')$ gives $M_i N_i$ as the double points of the involution $I(AA', BB')$. Therefore $|AA'B| = |A'AB'|$ which is an invariant C-curve and thus by Theorem 12 must pass through the points $M_j N_j$.

THEOREM 15. *If some C-curve is left invariant by a non-involutoric transformation π of G, then there is one and only one C-curve invariant under π through every point P of S not co-axial with a double point of π.*

Let C be the C-curve invariant under π, and let A be any point of C not a double point of π. Let π_1 be the transformation of G having the same double points as π and such that $\pi(A) = P$. Let $\pi_1(C) = C'$. Then

$$\pi_1 \pi \pi_1^{-1}(C') = C'$$

but $\pi_1 \pi \pi_1^{-1} = \pi$, therefore $\pi(C') = C'$, which proves that there is at least one C-curve through P invariant under π. Now suppose that there are two distinct C-curves through P invariant under π and let

$$\pi(P) = P', \qquad \pi(P') = P''.$$

The curve C' being invariant, gives P, P', P'' as points of C'. If $P'' \neq P$ these three points determine C' uniquely: if $P'' = P$, π is an involution and this contradicts the hypothesis.

THEOREM 16. *If the transformation π of G is parabolic (i. e., has only one double point) then through every point in S non-co-axial with the double point, passes one and only one C-curve invariant under π and every such curve passes through the double point.*

Let A be any point in S and $\pi(A) = A'$, $\pi(A') = A''$. Then, if M is the double point, (MA', AA'') is a harmonic set. But $\pi(AA'M) = A'A''M$. Therefore, there is a C-curve through A invariant under π. That there is only one follows from Theorem 15.

Tangent C-Curves. Two C-curves are said to be tangent to each other provided there exists a parabolic transformation π of G which leaves them

both invariant.* Two such C-curves will have only one point M in common (Theorem 16). The point M is called the point of contact.

The relation of tangency is clearly an invariant of the geometry $[S, G]$.

THEOREM 17. *If M is a point on a C-curve C and P any point non-co-axial with M and not on C, then there exists one and only one C-curve through P and tangent to C at M.*

For let π be any parabolic transformation of G with double point M, and let $\pi(P) = P'$. $|PP'M|$ is then the C-curve sought.

Orthogonal C-Curves. A pair of non-co-axial points M, N of S_3 determines uniquely three other pairs of non-co-axial points, namely the other three pairs of conjugate double points of the involution of which M, N are one pair of conjugate double points. In real S_3 if M, N are real, the other three pairs may all be imaginary. It seems convenient, therefore, to introduce the following terminology derived by obvious analogy from the geometry (S_2, G_2).

A C-curve C is said to be *about* two non-co-axial points M, N, if it passes through the points of one of the three pairs of non-co-axial points determined by M, N. The points M, N are then said to be *conjugate* with respect to C.

The relation here defined between a C-curve and certain pairs of points is clearly an invariant under G_3, as expressed by the following theorem:

THEOREM 18. *If a C-curve C is transformed by a transformation π of G into C', any pair of points conjugate with respect to C is transformed by π into a pair of points conjugate with respect to C'.*

A C-curve C_2 is said to be *orthogonal* to a C-curve C_1, if C_2 contains two points conjugate to C_1.†

THEOREM 19. *If C_2 is orthogonal to C_1, C_1 is orthogonal to C_2.*

---

* Two C-curves tangent to each other as here defined are tangent in the ordinary sense, and conversely. For the coördinate system described on p. 236 may be so chosen that any point M is $(0, 0, 0)$ and that any C-curve through M is given by $x = y = z$. Any other C-curve through M is then given by the equations

$$\frac{ax}{cx + d} = \frac{a'y}{c'y + d'} = \frac{a''z}{c''z + d''}, \qquad aa'a''dd'd'' \neq 0. \tag{1}$$

The conditions that these intersect in a second point $T = (t, t, t)$ $(t \neq 0)$ are

$$\frac{a}{ct + d} = \frac{a'}{c't + d'} = \frac{d''}{c''t + d''},$$

and that they be tangent in the ordinary sense at M $(t = 0)$ are

$$\frac{a}{d} = \frac{a'}{d'} = \frac{a''}{d''}.$$

But these are precisely the conditions that (1) can be obtained from $x = y = z$ by a parabolic transformation with double point M.

† It should be noted that orthogonality, as here defined, is not in general equivalent to orthogonality in the ordinary sense.

THEOREM 20. *Any pair of orthogonal C-curves is transformed by a transformation of G into a pair of orthogonal C-curves.*

This theorem establishes orthogonality as an invariant of G.

THEOREM 21. *If a transformation π of G leaves each of two orthogonal C-curves invariant, π is an involution.*

This follows from the proof of Theorem 15.

THEOREM 22. *Through two distinct points A, B on a C-curve C in S_3, pass three and only three C-curves orthogonal to C.*

For, let M, N be any pair of points on C harmonic with A, B: and let $M_i N_i$ $(i = 1, 2, 3)$ be the three other pairs of double points of the involution I having M, N for double points. Then we have $H(M_i N_i, AB)$ and the three curves $|ABM_i|$ $(i = 1, 2, 3)$ are orthogonal to C. That no other C-curves orthogonal to C pass through A, B will follow, if we show that the three C-curves orthogonal to C obtained as above from any other two points M', N' of C, harmonic with AB, coincide with those obtained from M, N. To this end we will show that if $ABMM'$ are on C, then $ABM_1 M_1'$ will be on the same C-curve. The following notational scheme is used:

$$M = (u_1, u_2, u_3), \qquad N = (v_1, v_2, v_3), \qquad M' = (u_1', u_2', u_3'),$$

$$N' = (v_1', v_2', v_3'), \qquad A = (\alpha_1, \alpha_2, \alpha_3), \qquad B = (\beta_1, \beta_2, \beta_3):$$

finally $M_1 = (v_1, u_2, u_3)$ and $M_1' = (v_1', u_2, u_3)$. The condition that A, B, M, M' are on C is

$$R_I(\alpha_1\beta_1, u_1 u_1') = R_I(\alpha_2\beta_2, u_2 u_2') = R_I(\alpha_3\beta_3, u_3 u_3').$$

We must show that

$$R_I(\alpha_1\beta_1, v_1 v_1') = R_I(\alpha_2\beta_2, u_2 u_2') = R_I(\alpha_3\beta_3, u_3 u_3')$$

or simply that

$$R_I(\alpha_1\beta_1, u_1 u_1') = R_I(\alpha_1\beta_1, v_1 v_1').$$

But this equality follows directly from the relations $H(\alpha_1\beta_1, u_1 v_1)$ and $H(\alpha_1\beta_1, u_1' v_1')$. The theorem is clearly valid in any geometry (S, G), if the number of C-curves through A, B orthogonal to C is replaced by $2^{n-1} - 1$.

Orthogonal Systems of C-Curves. It is now natural to generalize the well-known orthogonal systems of circles in (S_2, G_2). These are of two kinds. (1) The system of all circles through two distinct points A, B and the system of all circles about the two points A, B. (2) The system of all circles tangent at a point M to a given circle through M and the system of all circles tangent at M to a circle through M orthogonal to a circle of the first system.

In (S_3, G_3) we have analogously:

THEOREM 23. *Two non-co-axial points M, N of S_3 determine four systems of C-curves, viz., the system of all C-curves through M, N and the three systems of C-curves about M, N. Any C-curve of one of these four systems is orthogonal to every C-curve of the other three systems.*

THEOREM 24. *Given four mutually orthogonal C-curves C_1, C_2, C_3, C_4 through M, P. The four systems of C-curves consisting of all the C-curves tangent to C_i $(i = 1, 2, 3, 4)$ at M are such that any C-curve of one system is orthogonal to every C-curve of the other three systems.*

In (S_n, G_n), analogous theorems hold, the number of mutually orthogonal systems of C-curves being 2^{n-1}.

DARTMOUTH COLLEGE

A REDUCTION OF CERTAIN ANALYTIC DIFFERENTIAL EQUATIONS TO DIFFERENTIAL EQUATIONS OF AN ALGEBRAIC TYPE*

BY

WILLIA ! DUNCAN MACMILLAN

INTRODUCTION

The theorem that a system of power series in n dependent variables and p independent variables can be reduced to an equivalent system which are polynomials with respect to the dependent variables† leads rather naturally to the suggestion that possibly a corresponding reduction can be effected in the case of analytic differential equations. It might be anticipated that the theory for such a reduction of differential equations is a more complicated one than the corresponding theory for implicit functions. In certain respects this anticipation is amply justified, while in certain other respects, perhaps, it is not.

In effecting such a reduction many different methods and points of view are possible. The method adopted in the present paper is a substitution on the dependent variables of the form

$$x_i = \sum_{j=1}^{n} a_{ij} y_j + \text{terms of higher degree},$$

where the determinant $|a_{ij}|$ is distinct from zero. Such a substitution might well be called a *linear-transcendental* substitution.‡ It is certainly transcendental in general, and it partakes of the nature of a linear substitution in that, for small values of the variables, the correspondence between the variables x and y is unique. We seek then to determine a substitution of this type such that the resulting differential equations in the y's are algebraic.

Let the differential equations be of the form

$$\frac{dx_i}{dt} = X_i(x_1, \cdots, x_n) \qquad (i = 1, \cdots, n),$$

* Presented to the Society, December 26, 1913, and April 10, 1914.

† MacMillan, Mathematische Annalen, LXXII (1912), pp. 157–179.

‡ Since this paper was written (1911) there has appeared a paper by H. Dulac on the integrals of differential equations in which he uses a transformation which turns out to be a linear-transcendental substitution. His methods, however, are quite different from those employed here. See H. Dulac, Bulletin de la Société Mathématique de France, vol. 40 (1912), pp. 324–383.

in which the X_i are expansible in powers of $x_1, \cdots, x_n$ and vanish with these variables. In the present paper it will be supposed that the determinant of the linear terms of the X_i is distinct from zero, and that its "equation in s," formed by subtracting s from each term of the main diagonal of the determinant, is such that all of its roots, considered as points in the complex plane, lie on one side of a straight line passing through the origin and at a finite distance from it. No hypothesis is made with respect to the nature of the multiplicity of the roots. Under these conditions it will be shown that, aside from certain exceptional cases which are fully discussed, a substitution exists such that the differential equations in the y's are identical with the linear terms of the differential equations in the x's, and in the exceptional cases they are simply related to the linear terms. If the hypothesis that *all* of the roots of the equation in s lie on one side of a straight line through the origin is not satisfied then, unless $s = 0$ is a root of multiplicity n, one can always draw a line through the origin which will leave certain of the roots at a finite distance on one side, let us say p such roots. Then there exists a substitution which involves only p of the variables y such that the differential equations for the y's are linear. The solution for the x's in this event will not be complete for they involve only $p < n$ arbitrary constants.

Case I

1. Roots of the Equation in s are all Simple

Let us suppose that the differential equations are

$$\frac{d\xi_i}{dt} = \Xi_i(\xi_1, \cdots, \xi_n) \qquad (i = 1, \cdots, n), \tag{1}$$

where the Ξ_i are analytic functions of $\xi_1, \cdots, \xi_n$ which do not contain t. Let us suppose that these equations are satisfied by the constant solutions $\xi_i = \xi_i^{(0)}$. We wish to examine the solutions of (1) in the neighborhood of $\xi_i = \xi_i^{(0)}$. Accordingly we write $\xi_i = \xi_i^{(0)} + \xi_i^{(1)}$ and the differential equations become

$$\frac{d\xi_i^{(1)}}{dt} = A_{i1}\,\xi_1^{(1)} + \cdots + A_{in}\,\xi_n^{(1)} + \text{terms of higher degree}, \tag{2}$$

where the A_{ij} are constants.

Suppose now that the roots of the equation in s,

$$\begin{vmatrix} A_{11} - s, & A_{12}, & \cdots, & A_{1n} \\ A_{21}, & A_{22} - s, & \cdots, & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1}, & A_{n2}, & \cdots, & A_{nn} - s \end{vmatrix} = 0, \tag{3}$$

are all simple and distinct from zero. Let these roots be $\sigma_1, \cdots, \sigma_n$. Then, according to the theory of linear substitutions, there exists always a linear substitution

$$\xi_i^{(1)} = a_{i1}\, x_1 + a_{i2}\, x_2 + \cdots + a_{in}\, x_n,$$

the determinant of which is not zero, which transforms equations (2) into the canonical form

$$\frac{dx_i}{dt} = \sigma_i x_i + X_i \qquad (i = 1, \cdots, n), \tag{4}$$

where the X_i are power series in $x_1, \cdots, x_n$ of order not less than two; that is, the X_i do not contain any terms of degree less than two in $x_1, \cdots, x_n$.

We seek now a substitution

$$x_i = y_i + P_i(y_1, \cdots, y_n), \tag{5}$$

where the P_i are power series in $y_1, \cdots, y_n$ of order not less than two, say

$$P_i = \sum \alpha_{j_1 \cdots j_n}^{(i)} \cdots y_1^{j_1} \cdots y_n^{j_n} \qquad (j_1 + j_2 + \cdots + j_n \geqq 2), \tag{6}$$

which will simplify equations (4) as much as possible. The substitution of (5) in (4) gives rise to the equations

$$\begin{aligned}
&\left[1 + \frac{\partial P_1}{\partial y_1}\right]\frac{dy_1}{dt} + \frac{\partial P_1}{\partial y_2}\frac{dy_2}{dt} + \cdots + \frac{\partial P_1}{\partial y_n}\frac{dy_n}{dt} - \sigma_1 y_1 - \sigma_1 P_1 \\
&\qquad = Y_1(y_j + P_j), \\
&\frac{\partial P_2}{\partial y_1}\frac{dy_1}{dt} + \left[1 + \frac{\partial P_2}{\partial y_2}\right]\frac{dy_2}{dt} + \cdots + \frac{\partial P_2}{\partial y_n}\frac{dy_n}{dt} - \sigma_2 y_2 - \sigma_2 P_2 \\
&\qquad = Y_2(y_j + P_j), \\
&\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\frac{\partial P_n}{\partial y_1}\frac{dy_1}{dt} + \frac{\partial P_n}{\partial y_2}\frac{dy_2}{dt} + \cdots + \left[1 + \frac{\partial P_n}{\partial y_n}\right]\frac{dy_n}{dt} - \sigma_n y_n - \sigma_n P_n \\
&\qquad = Y_n(y_j + P_j).
\end{aligned} \tag{7}$$

Since these equations are linear in the dy_i/dt they can be solved for these quantities, but it is simpler to assume a solution for the dy_i/dt and then to examine whether equations (7) can be satisfied. Accordingly, we seek to determine the P_i so that the y_i shall satisfy the equations

$$\frac{dy_i}{dt} = \sigma_i y_i, \tag{8}$$

which are the same as the linear terms of (4).

With these values of the dy_i/dt equations (7) become

$$\sum_{j=1}^{n} \sigma_j y_j \frac{\partial P_i}{\partial y_j} - \sigma_i P_i = Y_i \qquad (i = 1, \cdots, n), \tag{9}$$

where the Y_i are the results of substituting (5) in the X_i. They are power series in $y_k + P_k$ $(k = 1, \cdots, n)$ of order not less than two in these quantities. Since the P_i are of order two in the y_k equations (9) can be satisfied. For let us substitute (6) in (9) and see whether the coefficients of the substitution, $\alpha^{(i)}_{j_1 \cdots j_n}$, can be determined. If we denote the expanded form of the right members by

$$Y_i = \sum \beta^{(i)}_{j_1 \cdots j_n} y_1^{j_1} \cdots y_n^{j_n},$$

we find, on comparing the right and left members of (9), that

$$[j_1 \sigma_1 + j_2 \sigma_2 + \cdots + j_n \sigma_n - \sigma_i] \alpha^{(i)}_{j_1 \cdots j_n} = \beta^{(i)}_{j_1 \cdots j_n}. \tag{10}$$

Let us denote $j_1 + j_2 + \cdots + j_n$ simply by j. Then for $j = 2$ the right members of (10) are known constants, and the quantities $\alpha^{(i)}_{j_1 \cdots j_n}$ are uniquely determined. When the $\alpha^{(i)}_{j_1 \cdots j_n}$ for $j = 2$ are known the $\beta^{(i)}_{j_1 \cdots j_n}$ for $j = 3$ are known; consequently the $\alpha^{(i)}_{j_1 \cdots j_n}$ for $j = 3$ are uniquely determined, and so on, sequentially. The process will fail if, and only if, for some set or sets of values $j_1, \cdots, j_n$ the coefficient

$$j_1 \sigma_1 + j_2 \sigma_2 + \cdots + j_n \sigma_n - \sigma_i = 0. \tag{11}$$

If the $\sigma_1, \cdots, \sigma_n$ are all real and of the same sign it can only happen exceptionally that (11) is satisfied. More generally, if the roots $\sigma_1, \cdots, \sigma_n$ considered as points in the complex plane lie all on one side of any straight line through the origin, it will be only exceptionally that expression (11) can be zero as was shown by Poincaré.*

Let us suppose that we are not dealing with one of the exceptional cases. Then we have formally determined a substitution (5) which transforms (4) into (8). Let us see whether this substitution converges. The solution of the equations

$$\sum (j_1 \sigma_1 + j_2 \sigma_2 + \cdots + j_n \sigma_n - \sigma_i) \alpha^{(i)}_{j_1 \cdots j_n} y_1^{j_1} \cdots y_n^{j_n} = \sum \beta^{(i)}_{j_1 \cdots j_n} y_1^{j_1} \cdots y_n^{j_n} = Y_i(y_j + P_j) \quad (i = 1, \cdots, n), \tag{12}$$

which are the same as (9), will be dominated by the solutions of the equations

$$\epsilon_i \sum \alpha^{(i)}_{j_1 \cdots j_n} y_1^{j_1} \cdots y_n^{j_n} = \bar{Y}_i(y_j + P_j) \qquad (i = 1, \cdots, n), \tag{13}$$

where the dash over the Y_i indicates that all of the coefficients in the expansion of the Y_i shall be given the positive sign, and ϵ_i is the smallest value of

$$|j_1 \sigma_1 + j_2 \sigma_2 + \cdots + j_n \sigma_n - \sigma_i|$$

for all $j_1, \cdots, j_n$. But equations (13) are the same as

* Thesis (1879).

$$\epsilon_i P_i = \overline{Y}_i(y_j + P_j). \tag{14}$$

Since the $\overline{Y}_i$ are of order not less than two in the arguments indicated, the solutions of (14) are convergent by the fundamental theorem of implicit functions, provided the $\overline{Y}_i(y_j + P_j)$ are convergent series. Hence the solutions of (9), which are the substitutions under discussion, are convergent.

This convergence proof, which is about as simple as a convergence proof can be, labors under the disadvantage that apparently it cannot be extended to more complicated cases. It will be useful therefore to give a second proof.

It will be no restriction of generality to suppose that the modulus of each coefficient in the expansions of the Y_i of equations (9) is less than $\nu_i > 0$, for if it were not true it would be easy to change the variables so that it would be true.

If the points σ_i in the complex plane are all at a finite distance from a straight line passing through the origin, and are all on one side of this straight line, then, if we are not dealing with one of the exceptional cases, there exists a positive number ϵ such that*

$$|j_1\sigma_1 + j_2\sigma_2 + \cdots + j_n\sigma_n - \sigma_i| > \epsilon|j_1 + j_2 + \cdots + j_n - 1|.$$

Consequently, if we take $\nu_i = \epsilon/n$ the solutions of (9) will be dominated by the solutions of

$$\epsilon\left[\sum_{k=1}^{n} y_k \frac{\partial Q_i}{\partial y_k} - Q_i\right] = \frac{\epsilon}{n}\frac{\left[\sum_{k=1}^{n}(y_k + Q_k)\right]^2}{1 - \sum_{k=1}^{n}(y_k + Q_k)}. \tag{15}$$

Since the Q_i are power series of order two it is clear that $Q_1 = Q_2 = \cdots = Q_n$. Taking then $y = \sum_{j=1}^{n} y_i$; $Q = \sum_{j=1}^{n} Q_j$, equations (15) reduce to

$$y\frac{dQ}{dy} - Q = \frac{(y+Q)^2}{1-(y+Q)}. \tag{16}$$

On integrating and choosing the constant of integration so that Q shall be of the second order in y, we get

$$Q = y(e^{(y+Q)} - 1), \tag{17}$$

or

$$Q = y^2 + \frac{3}{2}y^3 + \frac{8}{3}y^4 + \frac{125}{24}y^5 + \cdots.$$

Since the series (17) for Q as a power series in y are convergent, it follows that the $Q_i = Q/n$ are convergent, and consequently the solutions of (9) are convergent.

* See Picard: *Traité d'Analyse*, vol. III, p. 6.

It will be useful also to notice from (17) that

$$\frac{dQ}{dy} = \frac{e^{(y+Q)}}{1-(y+Q)} - 1$$

$$= (y+Q) + \left(1+\frac{1}{2!}\right)(y+Q)^2 + \left(1+\frac{1}{2!}+\frac{1}{3!}\right)(y+Q)^3 + \cdots, \tag{18}$$

and that the coefficients of this expansion are all less than 3. Likewise, in the expansion (17),

$$Q = y\left[(y+Q) + \frac{1}{2!}(y+Q)^2 + \frac{1}{3!}(y+Q)^3 + \cdots\right], \tag{19}$$

the coefficients of the right members are each less than or equal to unity.

We will lay aside now for a moment the hypothesis that it is possible to draw a straight line through the origin which will leave *all* of the points σ_i on one side of it. If the σ_i are all distinct and none of them zero, it will certainly be possible to draw a line which will leave at least one half of them on one side. Let us suppose that k of these roots $\sigma_1, \cdots, \sigma_k$ are thus situated, $k \geqq \frac{1}{2}n$.

In this event we can make, instead of (5), the substitution

$$\begin{aligned} x_i &= y_i + P_i(y_1, \cdots, y_k) && (i = 1, \cdots, k), \\ x_i &= 0 + P_i(y_1, \cdots, y_k) && (i = k+1, \cdots, n). \end{aligned} \tag{20}$$

Then the equations corresponding to (7) are

$$\begin{aligned}
&\left[1+\frac{\partial P_1}{\partial y_1}\right]y_1' + \frac{\partial P_1}{\partial y_2}y_2' + \cdots + \frac{\partial P_1}{\partial y_k}y_k' - \sigma_1 y_1 - \sigma_1 P_1 = Y_1, \\
&\frac{\partial P_2}{\partial y_1}y_1' + \left[1+\frac{\partial P_2}{\partial y_2}\right]y_2' + \cdots + \frac{\partial P_2}{\partial y_k}y_k' - \sigma_2 y_2 - \sigma_2 P_2 = Y_2, \\
&\cdots\cdots\cdots\cdots\cdots\cdots, \\
&\frac{\partial P_k}{\partial y_1}y_1' + \frac{\partial P_k}{\partial y_2}y_2' + \cdots + \left[1+\frac{\partial P_k}{\partial y_k}\right]y_k' - \sigma_k y_k - \sigma_k P_k = Y_k, \\
&\frac{\partial P_{k+1}}{\partial y_1}y_1' + \frac{\partial P_{k+1}}{\partial y_2}y_2' + \cdots + \frac{\partial P_{k+1}}{\partial y_k}y_k' - 0 - \sigma_{k+1} P_{k+1} = Y_{k+1}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots, \\
&\frac{\partial P_n}{\partial y_1}y_1' + \frac{\partial P_n}{\partial y_2}y_2' + \cdots + \frac{\partial P_n}{\partial y_k}y_k' - 0 - \sigma_n P_n = Y_n,
\end{aligned} \tag{21}$$

where

$$y_k' = \frac{dy_k}{dt}.$$

If now we take

$$P_i = \sum \alpha^{(i)}_{j_1} \dots {}_{j_k} y_1^{j_1} \cdots y_k^{j_k} \qquad (j_1 + \cdots + j_k \geqq 2),$$

and

$$y_i' = \sigma_i y_i \qquad (i = 1, \cdots, k), \tag{22}$$

and if the coefficients in the expansions of the right members be denoted by $\beta^{(i)}_{j_1} \dots {}_{j_k}$, we will obtain, on comparing coefficients,

$$[j_1\sigma_1 + j_2\sigma_2 + \cdots + j_k\sigma_k - \sigma_i]\, \alpha^{(i)}_{j_1} \dots {}_{j_k} = \beta^{(i)}_{j_1} \dots {}_{j_k}. \tag{23}$$

If then we exclude the exceptional cases, the $\alpha^{(i)}_{j_1} \dots {}_{j_k}$ can be determined sequentially, just as before.

The solutions of equations (22), and therefore the expressions (20) for the x_i, contain only k arbitrary constants. The solutions thus obtained for equations (4) are incomplete, as we lack $n - k$ constants of integration.*

Case II

2. The Equation in s has Multiple Roots

When the equation in s has multiple roots the differential equations (2) can no longer be reduced to the simple form (4). They can always, however, be reduced to the form

$$\frac{dx_i}{dt} = \sum_{j=1}^{i-1} a_{ij} x_j + \sigma_i x_i + X_i \qquad (i = 1, \cdots, n), \tag{24}$$

whatever be the nature of the multiplicity of the roots of (3), where the constants a_{ij} are either zero or unity, and for a fixed i every a_{ij} is zero except perhaps one.

On making the substitution (5), viz.,

$$x_i = y_i + P_i(y_1, \cdots, y_n),$$

equations (24) become

* Since the solutions of equations (8) and (22) are $y_i = c_i e^{\sigma_i t}$, it follows that the x_i are expansible in powers of $c_i e^{\sigma_i t}$. This result was stated by Poincaré in his thesis for the case in which all of the σ_i lie on one side of a straight line through the origin and the condition $\Sigma j_k \sigma_k - \sigma_i \neq 0$ is satisfied. The case in which only k of the roots lie on one side of the straight line was discussed by Picard (*Traité D'Analyse*, vol. III). Thus the results obtained in Case I above are equivalent to those of Poincaré and Picard, though the method is different.

$$
\begin{aligned}
&\left[1+\frac{\partial P_1}{\partial y_1}\right]y_1'+\frac{\partial P_1}{\partial y_2}y_2'+\cdots+\frac{\partial P_1}{\partial y_n}y_n'-\sigma_1 y_1-\sigma_1 P_1 = Y_1,\\
&\frac{\partial P_2}{\partial y_1}y_1'+\left[1+\frac{\partial P_2}{\partial y_2}\right]y_2'+\cdots+\frac{\partial P_2}{\partial y_n}y_n'-\sigma_2 y_2-\sigma_2 P_2\\
&\qquad = Y_2+a_{21}(y_1+P_1),\\
&\qquad\qquad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\frac{\partial P_n}{\partial y_1}y_1'+\frac{\partial P_n}{\partial y_2}y_2'+\cdots+\left[1+\frac{\partial P_n}{\partial y_n}\right]y_n'-\sigma_n y_n-\sigma_n P_n\\
&\qquad = Y_n+\sum_{j=1}^{n-1}a_{nj}(y_j+P_j).
\end{aligned}
\tag{25}
$$

If now we assume that (24) can be reduced to its linear terms, we will have

$$
y_i' = \sum_{j=1}^{i-1} a_{ij}\,y_j+\sigma_i\,y_i, \tag{26}
$$

and the partial differential equations (25) defining the P_i are

$$
\begin{aligned}
&\sigma_1 y_1\frac{\partial P_1}{\partial y_1}+[a_{21}y_1+\sigma_2 y_2]\frac{\partial P_1}{\partial y_2}+\cdots+\left(\sum_{j=1}^{n-1}a_{nj}y_j+\sigma_n y_n\right)\frac{\partial P_1}{\partial y_n}\\
&\qquad -\sigma_1 P_1 = Y_1,\\
&\sigma_1 y_1\frac{\partial P_2}{\partial y_1}+[a_{21}y_1+\sigma_2 y_2]\frac{\partial P_2}{\partial y_2}+\cdots+\left(\sum_{j=1}^{n-1}a_{nj}y_j+\sigma_n y_n\right)\frac{\partial P_2}{\partial y_n}\\
&\qquad -\sigma_2 P_2 = Y_2+a_{21}P_1,\\
&\qquad\qquad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\sigma_1 y_1\frac{\partial P_n}{\partial y_1}+[a_{21}y_1+\sigma_2 y_2]\frac{\partial P_n}{\partial y_2}+\cdots+\left(\sum_{j=1}^{n-1}a_{nj}y_j+\sigma_n y_n\right)\frac{\partial P_n}{\partial y_n}\\
&\qquad -\sigma_n P_n = Y_n+\sum_{j=1}^{n-1}a_{nj}P_j.
\end{aligned}
\tag{27}
$$

If now, bearing in mind the fact that the P_i are power series in $y_1, \cdots, y_n$ of order two, we write $P_i = \sum \alpha^{(i)}_{j_1\cdots j_n} y_1^{j_1}\cdots y_n^{j_n}$ and seek to determine the coefficients $\alpha^{(i)}_{j_1\cdots j_n}$ so as to satisfy (27), we find they are no longer determined by the simple formulæ (10), viz.,

$$
(j_1\sigma_1+j_2\sigma_2+\cdots+j_n\sigma_n-\sigma_i)\,\alpha^{(i)}_{j_1\cdots j_n}=\beta^{(i)}_{j_1\cdots j_n}.
$$

It is necessary to consider simultaneously the homogeneous terms (for each i) for which $j_1+j_2+\cdots+j_n = j$, $j = 2, \cdots \infty$. This leads to systems of linear equations in the $\alpha^{(i)}_{j_1\cdots j_n}$ which if suitably arranged will have a determinant in which the coefficients $(j_1\sigma_1+j_2\sigma_2+\cdots+j_n\sigma_n-\sigma_i)$ occur in

the main diagonal and all of the other elements of the determinant which are not zero lie on one side of the main diagonal. Hence the determinant is equal to

$$\Delta = \prod (j_1\sigma_1 + j_2\sigma_2 + \cdots + j_n\sigma_n - \sigma_i) \qquad (\Sigma j_k = j),$$

which can vanish if, and only if, for some set of values $j_1, \cdots, j_n$ we have

$$j_1\sigma_1 + j_2\sigma_2 + \cdots + j_n\sigma_n - \sigma_i = 0,$$

which is exactly the same condition (11) as in Case I.

Assuming that no such set of integers exists the coefficients $\alpha^{(i)}_{j_1 \cdots j_n}$ can be determined uniquely so that equations (27) are satisfied. If the series thus obtained are convergent the differential equations (24) are reduced to their linear terms (26).*

The convergence proof is not so simple as in Case I, but the second proof given there can be modified so as to fit the present situation.

On rearranging the left members of (27), we have

$$(27') \quad \left[\sum_{k=1}^{n}\sigma_k y_k \frac{\partial P_i}{\partial y_k} - \sigma_i P_i\right] + \sum_{k=2}^{n}\left(\sum_{j=1}^{k-1} a_{kj} y_j\right)\frac{\partial P_i}{\partial y_k} - \sum_{j=1}^{i-1} a_{ij} P_j = Y_i \qquad (i = 1, \cdots, n).$$

The solutions of these equations are dominated by the solutions of the equations

$$(28) \qquad \epsilon\left[\sum_{k=1}^{n} y_k \frac{\partial Q_i}{\partial y_k} - Q_i\right] - \sum_{k=2}^{n}\left(\sum_{j=1}^{k-1} y_j\right)\frac{\partial Q_i}{\partial y_k} - \sum_{j=1}^{i-1} Q_j = \overline{Y}_i,$$

provided the $\overline{Y}_i$ dominate the Y_i. Now make the transformations

$$(29) \qquad y_i = \mu^{n-i} z_i, \qquad Q_i = \mu^{n-i} R_i, \qquad \overline{Y}_i = \mu^{n-i} Z_i.$$

After removing the factor μ^{n-i} equations (28) become

$$(30) \quad \epsilon\left[\sum_{k=1}^{n} z_k \frac{\partial R_i}{\partial z_k} - R_i\right] - \sum_{k=2}^{n}\left(\sum_{j=1}^{k-1} \mu^{k-j} z_j\right)\frac{\partial R_i}{\partial z_k} - \sum_{j=1}^{i-1} \mu^{i-j} R_j = Z_i.$$

Obviously the solutions of (28) converge if the solutions of (30) converge.

For $\mu = 0$ equations (30) reduce to

$$(31) \qquad \epsilon\left[\sum_{k=1}^{n} z_k \frac{\partial R_i^{(0)}}{\partial z_k} - R_i^{(0)}\right] = Z_i^{(0)};$$

and if we take

$$Z_j^{(0)} = \frac{\epsilon}{n}\frac{[\sum (z_j + R_j^{(0)})]^2}{1 - \sum (z_j + R_j^{(0)})},$$

* The results obtained in Cases II and III are in agreement with those given by Dulac, *loc. cit.* though he did not give proofs of convergence.

equations (31) do not differ from (15) and therefore have the same solutions. From the properties of these solutions given in (18) and (19) it is easy to build a set of dominant functions Z_i for equations (30) for $\mu \neq 0$. Indeed, they will be so constructed that (30) and (31) have the same solutions. The equations which satisfy this condition are

$$\epsilon\left[\sum_{k=1}^{n} z_k \frac{\partial R_i}{\partial z_k} - R_i\right] - \sum_{k=2}^{n}\left(\sum_{j=1}^{k-1} \mu^{k-j} z_j\right)\frac{\partial R_i}{\partial z_k} - \sum_{j=1}^{i-1} \mu^{i-j} R_j$$

$$= \frac{\epsilon}{n}\frac{[\sum (z_j + R_j)]^2}{1 - \sum (z_j + R_j)} - \frac{1}{n}\sum_{k=2}^{n}\left(\sum_{j=1}^{k-1} \mu^{k-j} z_j\right)\left(\frac{e^{\sum (z_j + R_j)}}{1 - \sum (z_j + R_j)} - 1\right) \tag{32}$$

$$- \frac{1}{n}\sum_{k=1}^{i-1} \mu^{i-k}\left(\sum_{j=1}^{n} z_j\right) e^{\sum (z_j + R_j)} - 1).$$

If now we call $\sum_{k=1}^{n} \mu^k = m$, then the right members of these equations are expansible in powers of $\sum (z_j + R_j)$ with coefficients none of which are less than $(\epsilon - 4mn)/n$ which for sufficiently small values of μ is certainly positive, and the solutions of (32) are certainly convergent.

Returning now to equations (28) by means of (29), we see that the coefficients of $\bar{Y}_i$ are positive and not less than $(\epsilon - 4mn)/n\,\mu^{n-i}$. We have seen that without loss of generality the coefficients of the Y_i can be supposed less than $\nu_i > 0$. If then we take

$$\nu_i < \frac{\epsilon - 4mn}{n}\mu^n$$

the solutions of (30) will dominate the solutions of (28) which are accordingly convergent.

On combining the results of Cases I and II, we have the following theorem:

THEOREM. *If $dx_i/dt = X_i(x_1, \cdots, x_n)$ $(i = 1, \cdots, n)$ is a system of differential equations in which the X_i are analytic in $x_1, \cdots, x_n$, regular for $x_1 = \cdots = x_n = 0$ and vanish for these values of the variables; and if the equation in s, formed by subtracting s from the main diagonal of the functional determinant of the X_i for $x_1 = \cdots = x_n = 0$, has all of its roots $\sigma_1, \cdots, \sigma_n$ (considered as points in the complex plane) lying on one side of any straight line through the origin; and if there exists no set of integers $j_1, \cdots, j_n$ $(\sum_{k=1}^{n} j_k \geqq 2)$ such that $j_1\sigma_1 + j_2\sigma_2 + \cdots + j_n\sigma_n - \sigma_i = 0$, then there exists a linear-transcendental substitution*

$$x_i = \sum_{j=1}^{n} \alpha_{ij} y_i + \text{terms of higher degree}$$

such that the differential equations for the variables y are the same as the linear terms of the differential equations of the variables x, and such that the determinant $|\alpha_{ij}|$ is distinct from zero.

CASE III

3. EXCEPTIONAL CASES IN WHICH THERE EXIST SETS OF INTEGERS $j_1, \cdots, j_n$, SUCH THAT $\sum_{k=1}^{n} j_k \sigma_k - \sigma_i = 0$

We still retain the hypothesis that the points $\sigma_1, \cdots, \sigma_n$ lie on one side of a straight line through the origin. Notwithstanding this hypothesis it may happen that there exist sets of integers for which

$$j_1 \sigma_1 + j_2 \sigma_2 + \cdots + j_n \sigma_n - \sigma_i = 0.$$

From the preceding discussion it is clear that in this event it is not possible, in general, to reduce the differential equations to their linear terms. It is clear also that there can exist only a finite number of sets of integers for which the condition $\sum j_k \sigma_k - \sigma_i = 0$ is satisfied, for if we divide through by $\sum j_k = j$ we have

$$\frac{\sum j_k \sigma_k}{\sum j_k} = \frac{\sigma_i}{j}. \tag{33}$$

The left member is the expression for the center of gravity of masses equal to j_k situated at the points σ_k, and this center of gravity must lie within the polygon which encloses the points σ_k and cannot approach the origin. For increasingly large values of j the right member approaches zero. It is clear that (33) cannot be satisfied by any set of integers whose sum is sufficiently large.

Let us return now to equations (4), (5), and (6),

$$\frac{dx_i}{dt} = \sigma_i x_i + X_i, \tag{4}$$

$$x_i = y_i + P_i(y_1, \cdots, y_n), \tag{5}$$

$$P_i = \sum \alpha^{(i)}_{j_1 \cdots j_n} y_1^{j_1} \cdots y_n^{j_n} \qquad \left(\sum_{k=1}^{n} j_k \geqq 2\right). \tag{6}$$

Let us suppose that the maximum value of the sum, $j_1 + j_2 + \cdots + j_n$, for which $j_1 \sigma_1 + j_2 \sigma_2 + \cdots + j_n \sigma_n - \sigma_i = 0$ is $s - 1$.* Let us also assume the form of the differential equations for the y_i to be

$$\begin{aligned} \frac{dy_i}{dt} &= \sigma_i y_i + \sum{}^{*} b^{(i)}_{j_1 \cdots j_n} y_1^{j_1} \cdots y_n^{j_n} \qquad \left(\sum_{k=1}^{n} j_k \geqq 2\right), \\ &= \sigma_i y_i + p_i, \end{aligned} \tag{34}$$

where the terms included under the symbol $\sum^{*}$ are such that

$$j_1 \sigma_1 + j_2 \sigma_2 + \cdots + j_n \sigma_n - \sigma_i = 0.$$

* The s used here has no relation to the "equation in s" used previously.

If this expression for the y_i' be substituted in (7), we have for the determination of the P_i and the $b_{j_1}^{(i)} \cdots _{j_n}$ the equations

$$(35) \qquad \sum_{k=1}^{n} (\sigma_k y_k + p_k) \frac{\partial P_i}{\partial y_k} - \sigma_i P_i = Y_i - \sum{}^{*} b_{j_1}^{(i)} \cdots _{j_n} y_1^{j_1} \cdots y_n^{j_n}.$$

There is no difficulty in solving these equations, for when the coefficient of $\alpha_{j_1}^{(i)} \cdots _{j_n}$ vanishes the coefficient $b_{j_1}^{(i)} \cdots _{j_n}$ appears, and takes the place of $\alpha_{j_1}^{(i)} \cdots _{j_n}$. Thus all of the $b_{j_1}^{(i)} \cdots _{j_n}$ and all of the $\alpha_{j_1}^{(i)} \cdots _{j_n}$ are determined with the exception of those $\alpha_{j_1}^{(i)} \cdots _{j_n}$ for which $\sum_{k=1}^{n} j_k \sigma_k - \sigma_i = 0$, and these are left arbitrary. Let us denote these arbitraries by $A_{j_1}^{(i)} \cdots _{j_n}$ and suppose that $|A_{j_1}^{(i)} \cdots _{j_n}| < M$. Let us suppose further that the computation has been carried out so far that all of the $b_{j_1}^{(i)} \cdots _{j_n}$ have been determined. Then all of the terms in the substitution $x_i = y_i + \cdots$ are known up to, but not including, terms of degree s. We write therefore

$$(36) \qquad x_i = y_i + f_i(y_1, \cdots, y_n) + R_i(y_1, \cdots, y_n),$$

where the f_i are known polynomials of degree $s - 1$ and order two, and the R_i are power series of order s.

It is seen from (35) that the functions R_i are determined by the equations

$$(37) \qquad \sum_{k=1}^{n} (\sigma_k y_k + p_k) \frac{\partial R_i}{\partial y_k} - \sigma_i R_i = S_i(y_1, \cdots, y_n; R_1, \cdots, R_n),$$

where the S_i are power series of order s in the arguments indicated. Since the S_i are convergent power series of order s it is no restriction upon the generality to suppose that the moduli of the coefficients are all less than $\nu_i > 0$. If the positive constant a is properly chosen the solutions of equations (37) are dominated by the solutions of the equations

$$(38) \qquad \epsilon \left[\sum_{j=1}^{n} \left\{ y_j - \frac{a}{n}(y^2 + \cdots + y^{s-1}) \right\} \frac{\partial T_i}{\partial y_j} - T_i \right] = \frac{\epsilon}{n} \frac{(y+T)^s}{1-(y+T)} \qquad (i = 1, \cdots, n),$$

where for brevity we have taken $y = \sum_{k=1}^{n} y_k$, $T = \sum_{k=1}^{n} T_k$.

The solution of the system of equations (38) reduces, on taking the sum of all of the equations, to the single equation

$$(39) \qquad [y - a(y^2 + \cdots + y^{s-1})] \frac{dT}{dy} - T = \frac{(y+T)^s}{1-(y+T)}.$$

On substituting $T = v - y$, or $y + T = v$, we get

$$(40) \qquad [y - a(y^2 + \cdots + y^{s-1})] \frac{dv}{dy} - v = \frac{v^s}{1-v} - a(y^2 + \cdots + y^{s-1}).$$

The solution of equation (40) is dominated by the solution of

$$[y - a(y^2 + \cdots + y^{s-1})]\frac{dv}{dy} - v = \frac{v^s}{1-v}. \tag{41}$$

On integrating this equation and choosing the constant of integration so that $v = y + \cdots$, we get

$$\log\frac{v}{y} - \frac{1}{s-1}v^{s-1} - \cdots = a(y + \tfrac{1}{2}y^2 + \cdots + \frac{1}{s-2}y^{s-2}) + \cdots. \tag{42}$$

If now we set $v = (1+w)y$ it is seen that (42) admits a unique solution for w as a power series in y vanishing with y. Since this series is convergent and dominates the solutions of (37), it follows that the solutions of (37) are convergent. We have then

THEOREM II. *If, in the system of differential equations*

$$\frac{dx_i}{dt} = \sigma_i x_i + X_i(x_1, \cdots, x_n) \qquad (i = 1, \cdots, n),$$

where the X_i are convergent power series in $x_1, \cdots, x_n$ of order not less than two, the σ_i, considered as points in the complex plane, lie all on one side of a straight line passing through the origin, then there exists a convergent substitution

$$x_i = y_i + P_i(y_1, \cdots, y_n) \qquad (i = 1, \cdots, n),$$

where the P_i are power series in $y_1, \cdots, y_n$ of order not less than two, such that the variables y_i satisfy the differential equations

$$\frac{dy_i}{dt} = \sigma_i y_i + \sum{}^{*} b^{(i)}_{j_1 \cdots j_n} y_1^{j_1} \cdots y_n^{j_n} \qquad \Big(\sum_{k=1}^{n} j_k \geqq 2\Big), \tag{43}$$

where the terms included under the symbol Σ^* include all terms for which

$$j_1\sigma_1 + j_2\sigma_2 + \cdots + j_n\sigma_n - \sigma_i = 0,$$

and where the $b^{(i)}_{j_1 \cdots j_n}$ are properly chosen constants.

Notwithstanding the fact that equations (43) are non-linear they are easily integrated as is illustrated by the following example:

Let us suppose that $\sigma_1 = 1$, $\sigma_2 = 2$, and $\sigma_3 = 3$. Then the following relations exist

$$2\sigma_1 - \sigma_2 = 0, \qquad \sigma_1 + \sigma_2 - \sigma_3 = 0, \qquad 3\sigma_1 - \sigma_3 = 0,$$

and there are no others. Equations (43) become

$$\begin{aligned} y_1' &= y_1, \\ y_2' &= 2y_2 + Ay_1^2, \\ y_3' &= 3y_3 + By_1y_2 + Cy_1^3, \end{aligned} \tag{44}$$

where A, B, and C are certain constants. The solutions of equations (44) are readily found to be

$$y_1 = c_1 e^t,$$

$$y_2 = [c_2 + c_1^2 At] e^{2t},$$

$$y_3 = [c_3 + (c_1 c_2 B + c_1 C) t + \tfrac{1}{2} c_1^3 ABt^2] e^{3t}.$$

The solutions of these equations are analogous to the solutions of non-homogeneous linear differential equations in which the non-homogeneous terms have the same period as the homogeneous terms.

UNIVERSITY OF CHICAGO,
May, 1914

A NEW CANONICAL FORM OF THE ELLIPTIC INTEGRAL*

BY

BESSIE I. MILLER

INTRODUCTION

The elliptic norm curve Q_n in space S_{n-1} admits a group G_{2n^2} of collineations, and in fact there is a single infinity of such curves which admit the same group. A particular Q_n of the family is distinguished by a value of the parameter τ, itself an elliptic modular function defined by the modular group congruent to identity (mod. n).

In the group G_{2n^2}, there are certain involutory collineations with two fixed spaces. If Q_n is projected from one fixed space upon the other, a family of rational curves R_m mapping the family of Q_n's is obtained. The quadratic irrationality separating involutory pairs on Q_n involves the modulus τ, and the parameter t of the R_m. When the genus of the modular group is zero and $n = 3, 4, 5$, the irrationality can be used to define the elliptic parameter

$$u = \int \frac{(t\,dt)}{\sqrt{(t\tau)\,\alpha_\tau^{r-3}\,\alpha_t^3}},$$

where α_t^r is the tetrahedral, octahedral, or icosahedral form. This is in contrast with Klein's form as developed by Bianchi, for there the normal elliptic integral is a rational curvilinear integral taken along an elliptic curve.

A comparison of the two integrals is more illuminating if carried out for a special case. Let Q_n be Q_5 in S_4. Let $\phi_i(x_j, a) = 0$, $(i, j = 0, \cdots 4)$ be five quadrics intersecting in Q_5, where according to Bianchi's notation x_j are the variables, a the modulus. After a suitable transformation of coördinates the icosahedral form which appears in the irrationality is

$$\alpha_t^{12} = t_1\,t_2\,(t_1^{10} + 11t_1^5\,t_2^5 - t_2^{10}).$$

The integral u involving $\tau \equiv a$ explicitly in a rather simple form is uniquely defined. Moreover it is invariant under all cogredient transformations of t and τ, which leave α_x^{12} unaltered, i. e., the sixty transformations of the icosahedral group applied simultaneously to t and τ, the parameter of the doubly covered conic R_2 and the modulus of Q_5 respectively, leave u unaltered.

Consider now Bianchi's integral, defined as

* Presented to the Society, December 27, 1915.

$$U = C \int \frac{(w\,dv - v\,dw)}{(\phi_0\,\phi_1\,\phi_2\,w\,v)},$$

where C is a constant, w and v any two expressions linear in x_j, and the denominator is the functional determinant of ϕ_0, ϕ_1, ϕ_2, w, v. Different expressions for U can be obtained by making different choices for w and v. Hence there is no unique form for U. Under the G_{50} of collineations on x_j and under the transformations of $a \equiv \tau$, U assumes various conjugate forms.

Hence u has the advantage over U in simplicity and uniqueness of form, and also in its invariancy under transformations.

In section I by a study of the integral u some interesting results are obtained. The modular equation connecting τ and J, the absolute invariant of u, can be deduced as the result of the binary syzygy of lowest weight connecting the concomitants of α_x^r. The requirement that the Riemann surface attached to the modular equation be regular leads to the modular equations associated with the regular bodies. It is then possible to eliminate the more tedious individual proofs used by Bianchi in the discussion of the moduli of Q_3 and Q_5 to show that these moduli are the tetrahedral and icosahedral irrationalities respectively. In fact the algebraic discussion carried out once for α_x^r is complete for factor groups of genus zero, which have been discussed by Klein, i. e., those isomorphic with the groups associated with the regular bodies, namely the one dihedral group G_6, and the tetrahedral, octahedral, and icosahedral groups.

In sections II, III, IV, it is shown that the canonical form u occurs naturally in connection with the elliptic norm curves Q_3, Q_4, Q_5 and in section III the possibility of extending the canonical form to larger values of the genus is indicated.

Whenever possible the notation conforms to that of Klein-Fricke, Bianchi, and Grace and Young.*

I. A SYZYGY

1. Consider the elliptic integral of the first kind in the form given in the introduction.† The form under the radical sign, since it is a quartic in t,

* Abbreviations and references:

K. F. *Vorlesungen über die Theorie der elliptischen Modulfunctionen*, by Klein and Fricke.

B. *Über die Modulformen dritter und fünfter Stufe des elliptischen Integrals erster Gattung*, by Bianchi in Mathematische Annalen, vol. 17 (1880), pp. 234–262.

G. & Y. *Algebra of Invariants*, by Grace and Young.

† The integral u can be reduced to Weierstrass's canonical form by means of the transformation $x_1 = \alpha_\tau^{n-1}\,\alpha_t$, $x_2 = (t\tau)$. (Vide Clebsch, *Theorie der binären algebraischen Formen*, §§ 81–83, for similar transformations.)

has two invariants* g_2 and g_3, the former of degree $2(n-2)$ in τ, the latter of degree $3(n-2)$ in τ. But $f \equiv \alpha_\tau^n$ has one covariant, H, of degree $2(n-2)$ in τ, and one covariant, $(f, H)^1$, of degree $3(n-2)$ in τ. Hence g_2 must be some multiple of H; g_3 some multiple of $(f, H)^1$. The discriminant, Δ, of a quartic can be expressed in terms of g_2^3 and g_3^2. The absolute invariant J is then defined by the syzygy

$$J : J-1 : : 1 : : g_2^3 : 27 g_3^2 : \Delta .$$

In order to obtain an analogous syzygy involving the transvectants of f, it is necessary to obtain the expressions for g_2, g_3, and Δ, in terms of the transvectants of f and to determine some relation between these forms.

2. Let

$$f \equiv \alpha_\tau^n \equiv \beta_\tau^n \equiv \gamma_\tau^n \equiv \text{etc.},$$

$$g \equiv \alpha_\tau^{n-3}\, \alpha_t^3 (t\tau) \equiv \beta_\tau^{n-3}\, \beta_t^3 (t\tau) \equiv \text{etc.},$$

$$H \equiv (\alpha\beta)^2\, \alpha_\tau^{n-2}\, \beta_\tau^{n-2} = (f, f)^2,$$

$$H_g \equiv (g, g)^2,$$

$$\tilde{t} \equiv (\alpha\beta)^2 (\gamma\alpha)\, \alpha_\tau^{n-3}\, \beta_\tau^{n-2}\, \gamma_\tau^{n-1} = (f, H)^1,$$

$$i \equiv (\alpha\beta)^4\, \alpha_\tau^{n-4}\, \beta_\tau^{n-4} = (f, f)^4,$$

$$j \equiv (\alpha\beta)^2 (\beta\gamma)^2 (\gamma\alpha)^2\, \alpha_\tau^{n-4}\, \beta_\tau^{n-4}\, \gamma_\tau^{n-4} = (f, i)^2,$$

$$k \equiv (\alpha\beta)^6\, \alpha_\tau^{n-6}\, \beta_\tau^{n-6} = (f, f)^6.$$

According to Klein $(g, g)^4$ equals $2g_2$ and $(H_g, g)^4$ equals $4g_3$.

$$(g, g)^4 = \alpha_\tau^{n-3}\, \beta_\tau^{n-3} \left(\alpha_\tau^3 (t\tau), \beta_\tau^3 (t\tau) \right)^1 = -\frac{18}{4!} (\alpha\beta)^2\, \alpha_\tau^{n-2}\, \beta_\tau^{n-2}.$$

$$\therefore\ g_2 = -\tfrac{3}{8} H.$$

$$H_g\dagger = \tfrac{1}{2} (\alpha\beta)^2\, \alpha_\tau^{n-3}\, \beta_\tau^{n-3}\, \alpha_t \beta_t (t\tau)^2,$$

$$(H_g, g)^4 = \frac{6}{4!} (\alpha\beta)^2 (\gamma\alpha)\, \alpha_\tau^{n-3}\, \beta_\tau^{n-2}\, \gamma_\tau^{n-1}.$$

$$\therefore\ g_3 = \tfrac{1}{16} (f, H)^1 = \tfrac{1}{16} \tilde{t}.$$

Now $\tilde{t}$ is a jacobian. Therefore its square‡ is reducible. In fact

$$-2\tilde{t}^2 - H^3 = (H, H)^2 f^2 - 2 (f, H)^2 fH. \tag{1}$$

Let $\Delta' = 2\tilde{t}^2 + H^3$ and $J = H^3/\Delta'$. Then $J - 1 = -2\tilde{t}^2/\Delta'$ and the following proportion can be written

* K. F., I, pp. 13–15.

† G. & Y., §§ 43, 49, for methods of calculating transvectants.

‡ G. & Y., § 78.

$$J : J - 1 : 1 = H^3 : -2\tilde{t}^2 : \Delta'. \tag{2}$$

This is assuredly a relation between the hessian and jacobian of an $n$ic, similar to the one existing between the invariants, g_2 and g_3, of a quartic. Corresponding to the discriminant Δ of the quartic is the form Δ', expressed by (1) in terms of transvectants which are reducible. Hence before (2) is discussed, the reduction of $(H, H)^2$ and of $(f, H)^2$ is made.

3. The reduction just mentioned can be outlined as follows:

$$\begin{aligned}
(f, H)^2 &= \frac{1}{2n-5}(\alpha\beta)^2\{(n-3)(\alpha\gamma)^2\alpha_\tau^{n-4}\beta_\tau^{n-2}\gamma_\tau^{n-2} \\
&\quad + (n-2)(\alpha\gamma)(\beta\gamma)\alpha_\tau^{n-3}\beta_\tau^{n-3}\gamma_\tau^{n-2}\}^* \\
&= (\alpha\beta)^2(\alpha\gamma)^2\alpha_\tau^{n-4}\beta_\tau^{n-2}\gamma_\tau^{n-2} - \frac{n-2}{2(2n-5)}\,if \\
&= (\alpha\beta)^2(\alpha\gamma)^2\beta_\tau^2\gamma_\tau^2\alpha_\tau^{n-4}\beta_\tau^{n-4}\gamma_\tau^{n-4} - \frac{n-2}{2(2n-5)}\,if\,\dagger \\
&= \tfrac{1}{2}\,if - 2(\beta\gamma)^2(\alpha\beta)(\alpha\gamma)\alpha_\tau^{n-2}\beta_\tau^{n-3}\gamma_\tau^{n-3} - \frac{n-2}{2(2n-5)}\,if\,\ddagger \\
&= \frac{n-3}{2(2n-5)}\,if.
\end{aligned}$$

The reduction of $(H, H)^2$ is accomplished by means of Gordan's series.§ Since $(H, H)^2 = ((f, f)^2, H)^2$, the first series used is

$$\begin{pmatrix} f & f & H \\ n & n & 2n-4 \\ 0 & 2 & 2 \end{pmatrix}.$$

This gives

$$\begin{aligned}
(H, H)^2 + \frac{n-2}{2(2n-5)}\,iH &= ((f, H)^2, f)^2 + \frac{4(n-3)}{3n-8}((f, H)^3, f)^1 \\
&\quad + \frac{2(2n-7)}{3(3n-10)}\big((f, H)^4, f\big)^0.
\end{aligned} \tag{3}$$

The terms on the right-hand side of (3) must be reduced in turn so that (3) may contain only the transvectants enumerated in § 2. It is found that

* G. & Y., § 22. Apply identity III.

† Square identity III (loc. cit.) and apply it.

‡ G. & Y., § 22. Rewrite identity I thus: $(\beta\gamma)\alpha_\tau = (\beta\alpha)\gamma_\tau - (\gamma\alpha)\beta_\tau$ and multiply by $(\beta\gamma)(\alpha\beta)(\alpha\gamma)$ before applying.

§ G. & Y., § 54.

$$((f, H)^2, f)^2 = \frac{1}{2(2n-5)(3n-8)}\left\{n(n-3)iH - \frac{(2n-8)(2n-9)}{3}jf + \frac{(n-4)(n-5)}{3}kf^2\right\}.$$

The transvectant $(f, H)^3$ is given by the series

$$\begin{pmatrix} f & f & f \\ n & n & n \\ 0 & 2 & 3 \end{pmatrix},$$

and equals

$$\frac{n-4}{2(2n-5)}(f, i)^1,$$

$$((f, H)^3, f)^1 = \left(\frac{n-4}{2(2n-5)}(f, i)^1, f\right)^{1*}$$

$$= \frac{n-4}{2(2n-5)}\left\{-\frac{2n-9}{3n-10}\left[-j + \frac{n-5}{2(2n-9)}kf\right] + \tfrac{1}{2}iH\right\}.$$

The transvectant $(f, H)^4$ is given by the series

$$\begin{pmatrix} f & f & f \\ n & n & n \\ 0 & 4 & 2 \end{pmatrix}.$$

$$\therefore (f, H)^4 = -\frac{n-1}{2n-5}(i, f)^2 + \frac{(3n-10)(n-5)}{4(2n-7)(2n-9)}kf$$

$$= \frac{n-1}{2n-5}j + \frac{(n-4)(n-5)}{4(2n-7)(2n-9)}kf.$$

If the values of the different transvectants are substituted in (3), the following result is obtained:

$$(H, H)^2 = -\frac{1}{2(2n-5)}iH + \tfrac{1}{3}jf.$$

It is interesting to notice that the term kf^2 has the coefficient zero. Therefore k, a form which does not exist in the case of the quartic, does not appear in the final result.

4. The relation between t and H is given by (1). If now the results of §§ 2 and 3 are used, (1) can be replaced by

$$2\tilde{t}^3 + H^2 = f^2(\tfrac{1}{2}iH - \tfrac{1}{3}jf).$$

Then (2) can be written as follows:

* See G. & Y., § 77 (XIX), for the formula used to obtain the result given above.

$$J : J - 1 : 1 = 6H^3 : -12\tilde{t}^2 : f^2(3iH - 2jf). \tag{4}$$

If the equivalents of H and $\tilde{t}$ (vide § 2) are introduced in (4), then the usual form of the syzygy existing between the invariants of the quartic is obtained. Thus (4) is the desired syzygy between the hessian and jacobian of an $n$ic. It shows that, as H and $\tilde{t}$ correspond to g_2 and g_3 respectively, Δ' corresponds to Δ.

5. The syzygy (4) containing the parameter τ and the invariant J is for a given J an algebraic equation of degree $6(n-2)$, with roots which are algebraic functions of J. It can be mapped on a $6(n-2)$-sheeted Riemann surface. The k-fold roots of (4) and the values of J giving the branch points of the surface must be determined.

Since a k-fold root of (4) is a $(k-1)$-fold root of the transvectant $(H^3, \tilde{t}^2)^1$, the reduction of this transvectant is necessary. Let

$$H \equiv (\alpha\beta)^2 \alpha_\tau^{n-2} \beta_\tau^{n-2} \equiv h_\tau^{2(n-2)},$$

$$\tilde{t} \equiv (\delta\epsilon)^2 (\eta\delta) \delta_\tau^{n-3} \epsilon_\tau^{n-2} \eta_\tau^{n-1} \equiv t_\tau^{3(n-2)}.$$

Then

$$\begin{aligned}(H^3, \tilde{t}^2)^1 = \frac{H^2\tilde{t}(\alpha\beta)^2(\delta\epsilon)^2(\eta\delta)}{6(n-2)} \alpha_\tau^{n-3} \beta_\tau^{n-3} \{ &(n-3)(\alpha\delta)\beta_\tau \delta_\tau^{n-4} \epsilon_\tau^{n-2} \eta_\tau^{n-1} \\ &+ (n-2)(\alpha\epsilon)\beta_\tau \delta_\tau^{n-3} \epsilon_\tau^{n-3} \eta_\tau^{n-1} + (n-1)(\alpha\eta)\beta_\tau \delta_\tau^{n-3} \epsilon_\tau^{n-2} \eta_\tau^{n-2} \\ &+ (n-3)(\beta\delta)\alpha_\tau \delta_\tau^{n-4} \epsilon_\tau^{n-2} \eta_\tau^{n-1} + (n-2)(\beta\epsilon)\alpha_\tau \delta_\tau^{n-3} \epsilon_\tau^{n-3} \eta_\tau^{n-1} \\ &+ (n-1)(\beta\eta)\alpha_\tau \delta_\tau^{n-3} \epsilon_\tau^{n-2} \eta_\tau^{n-2} \} = H^2\tilde{t}(H, \tilde{t})^1 \\ = H^2\tilde{t}f \left\{ \frac{n-3}{3(n-2)} jf - \frac{2n-3}{3(n-2)} iH \right\}&.\end{aligned}$$

It is now evident that (4) has

$2(n-2)$ triple roots—the values for which $H = 0$;

$3(n-2)$ double roots—the values for which $\tilde{t} = 0$;

n double roots—the values for which $f = 0$;

$4(n-3)$ double roots—the values for which $(n-3)jf - (2n-3)iH = 0$.

The values of J giving the branch points of the Riemann surface are determined from the two following equations which are used instead of (4):

$$Jf^2(3iH - 2jf) = 6H^3, \tag{4a}$$

$$(J-1)(3iH - 2jf)f^2 = -12\tilde{t}^2. \tag{4b}$$

Then if

$$H = 0, \qquad J = 0;$$

$$\tilde{t} = 0, \qquad J = 1;$$

$$f = 0, \qquad J = \infty;$$

$$(n-3)jf - (2n-3)iH = 0,$$

$$J = \frac{6(3-n)H^2}{(n+3)if^2} = \frac{6(3-n)^3 j^2}{(n+3)(2n-3)^2 i^3}.$$

The form of the equations (4*a*) and (4*b*) shows that all the sheets of the $6(n-2)$-sheeted Riemann surface come together, k at a time, over the branch points $J = 0$ and $J = 1$ but do not do so over the remaining branch points.

6. The condition for the regularity of the Riemann surface at $J = \infty$ can be determined. The desirability of applying such a condition is obvious when the following theorem has been proved: *If the Riemann surface is regular over $J = \infty$, τ is an elliptic modular function.*

From (4) J is a uniform function of τ and of ω. By hypothesis the only critical points occur at $J = 0, 1, \infty$. But these give the vertices of the two nets of triangles obtained by mapping the real J axis on the τ and ω planes respectively.* The positive half of the J plane is mapped into a region within a triangle having one vertex at ∞ and the angles at the finite vertices equal to $\pi/2$ and $\pi/3$ respectively. This is true in both the τ and ω planes. Since a circuit around $J = 0$ necessitates $\omega(0) = \omega_0$† and $\tau(0) = \tau_0$‡ the two power series for J as a function of τ and of ω are

$$J = a_0(\omega - \omega_0)^3\{1 + P(\omega - \omega_0)\},$$

$$J = b_0(\tau - \tau_0)^3\{1 + P(\tau - \tau_0)\}.$$

Therefore τ is a uniform function of ω at ω_0; in fact

$$\tau - \tau_0 = \sqrt[3]{\frac{a_0}{b_0}}(\omega - \omega_0)\{1 + P(\omega - \omega_0)\}.$$

In like manner it is a uniform function of ω when $J = 1$. The first term of the series will be of the form

$$\tau - \tau_1 = \sqrt{\frac{a_1}{b_1}}(\omega - \omega_1),$$

where τ_1 and ω_1 are the values of τ and ω respectively for $J = 1$. A circuit around $J = \infty$ leads into the negative half-plane. This is not mapped in

* K. F., I, p. 45 ff.

† K. F., I, p. 128.

‡ (4).

the given region of the τ and ω plane but in the reflection of that region. Therefore in the map of the positive or negative half of the J plane, τ is a uniform function of ω and the theorem is proved.

But the condition that the Riemann surface be regular over $J = \infty$ is not yet known. In order to determine when all the sheets come together k at a time, assume that f and H have no common roots and hence that f has no double root. The form of $(H^2, \tilde{t})^1$ shows that $f = 0$ accounts for $2n$ sheets. Hence there are $6(n-2) - 2n = 4(n-3)$ sheets to be brought together over $J = \infty$ for the roots of $(n-3)jf - (2n-3)iH = 0$. But the corresponding form of J shows that only when i or f vanishes can $J = \infty$.

Let the values $(\mu_\tau^p)^k \equiv p^k = 0$ be those not included by $f = 0$, which give the branch point at $J = \infty$. The order of the branch point is then $k - 1$. Since the surface is to be regular

$$3iH - 2jf \backsim f^{k-2} p^k, \tag{5}$$

and

$$(n-3)jf - (2n-3)iH \backsim f^{k-2} p^{k-1} q,$$

where q refers to any other possible branch points. Since p is to determine $J = \infty$,

$$i \backsim ph,$$

where h is the result of dividing i by p. Consider the order in the variables on each side of (5). Then

$$4(n-3) = n(k-2) + kp, \tag{6}$$

where p indicates the order of the form p. If $k = 2$, $p = 2(n-3)$. Therefore the form p cannot be a factor of i whose degree is $2(n-4)$, unless i vanishes identically. If $k = 3$, since H by hypothesis cannot contain f as a factor, i must do so, because of (5); i. e., $i \backsim fph'$ where h' is the result of dividing h by f. But when $k = 3$, from (6) $p = n - 4$. Hence

$$2(n-4) = n + n - 4 + h',$$

where h' indicates the order of the form h'. This is obviously impossible. In fact since iH contains a factor f^{k-2} (vide (5)), and H contains no factor f, i must contain the factor f^{k-2}. Therefore

$$2(n-4) = p + n(k-2) + h'.$$

If $k > 3$, $p + h' \geqq -8$, again an impossibility, unless $i \equiv 0$. If $i \equiv 0$, then from (4a) $J = \infty$ when $jf^3 = 0$. But in the case under discussion J can equal infinity only when i or f vanishes. Hence j is either a constant or some power, l, of f; i. e.,

* The symbol $\backsim$ is used to indicate equivalence of forms to within a multiplicative constant.

$$3(n-4) = ln$$

or

$$n(3-l) = 12.$$

If $l = 0, 1, 2$, in turn, $n = 4, 6, 12$. For the *nics*, where n assumes only these values, the tetrahedral, octahedral, and icosahedral forms are obtained, when the condition $i \equiv 0$ is applied. On the other hand it is known* that in the case of the elliptic modular equations, the branch points of the Riemann surface can occur only at $J = 0, 1, \infty$.

The conclusions of the preceding paragraphs can be summarized as follows: *If the Riemann surface on which* (4) *is mapped is made regular, then* $i \equiv 0$, τ *is an elliptic modular function,* f *is a form associated with one of the regular bodies, and* (4) *is the corresponding modular equation.*

The dihedral case has not yet been shown to be included in (4) but can easily be developed. Let

$$f \equiv (\alpha_1 \tau_1 + \alpha_2 \tau_2)^n \equiv \tau_1^n - \tau_2^n.$$

Then

$$H = -2\tau_1^{n-2}\tau_2^{n-2},$$

$$\tilde{t} = -\tau_1^{n-3}\tau_2^{n-3}(\tau_1^n + \tau_2^n),$$

$$i = -2\tau_1^{n-4}\tau_2^{n-4},$$

$$j \equiv 0.$$

The syzygy (4) becomes

$$J : J - 1 : 1 = -4\tau_1^n \tau_2^n : -(\tau_1^n + \tau_2^n)^2 : (\tau_1^n - \tau_2^n)^2.$$

This can easily be identified with the form of the dihedral equation given by Klein,† if the substitution $J = 1/z$ is made.

II. The elliptic norm curve of third order, Q_3, in two-dimensional space, S_2

7. Since the general configuration of the elliptic norm curve of third order Q_3 in two-dimensional space S_2, is familiar, it will be discussed in this paper only as it offers an opportunity of giving a geometric significance to some of the algebraic results previously obtained.

According to the work of Klein and Bianchi, Q_3, whose three homogeneous coördinates can be expressed as products of sigma functions, is transformed into itself by the group G_{18} of collineations which send the space S_2 into itself.‡ The normal form of its equation can be written

$$x_0^3 + x_1^3 + x_2^3 - 3\tau x_0 x_1 x_2 = 0. \tag{7}$$

* K. F., I, p. 574, § 1.

† Klein, *Vorlesungen über das Ikosaeder*, p. 60.

‡ K. F., II, pp. 246–256.

In the coördinate system used, the reference triangle is an inflexional triangle. Since τ is defined by (7) in terms of a ratio of the x's, τ is an elliptic modular function, i. e., a function of ω_1/ω_2, where ω_1 and ω_2 are the periods of the sigma functions in terms of which x_i is expressed.*

If in (7) τ is considered a parameter, then (7) represents a pencil of cubics on the nine inflexions. The degenerate curves of the pencil are the inflexional triangles and they intersect the proper curves at the inflexions of the proper curves.

8. An irrationality of the type discussed in section I occurs if one of the involutory† collineations in G_{18} is considered. One such collineation is $x_0(-u) = x_0(u)$, $x_1(-u) = x_2(u)$, $x_2(-u) = x_1(u)$, where the center is the inflexion $(0, 1, -1)$ and the axis upon which Q_3 is projected is the harmonic polar $x_1 - x_2 = 0$. A pencil of lines

$$t_1 x_0 + t_2(x_1 + x_2) = 0 \tag{8}$$

on the center establishes a two-to-one correspondence between the points of Q_3 and the points on the axis. The irrationality separating two corresponding points of Q_3 is from (7) and (8) found to be

$$\frac{x_1}{x_2} = \frac{t^3 - 3\tau t^2 + 2 \pm \sqrt{-3t^2(t^4 + 2\tau t^3 - 3\tau^2 t^2 - 4t + 4\tau)}}{2(t^3 - 1)} \qquad (t = t_1 \text{ or } t_2).$$

Let

$$t^4 + 2\tau t^3 - 3\tau^2 t^2 - 4t + 4\tau = (t - \tau)(t^3 + 3\tau t^2 - 4) \equiv a_t^4.$$

The values of t for which $a_t^4 = 0$ give the double points of the involution, which are the points of contact of the tangents drawn from the inflexion. Since one of these tangents is the flex tangent which has the parameter $t = \tau$, a_t^4 can be written in the factored form given. The four flex lines on the point $(0, 1, -1)$ have parameters given by

$$\alpha_t^4 \equiv t_2(t_1^3 - t_2^3) = 0,‡$$

an equianharmonic quartic. If α_t^4 is polarized with respect to τ, it becomes evident that $a_t^4 = (t\tau)\alpha_\tau \alpha_t^3$. Hence the irrationality $\sqrt{a_t^4}$ is of exactly the type given in section I. A geometric interpretation of the result is that *at an inflexion I of Q_3, the three tangents from I other than the flex tangent are given by the first polar of the flex tangent at I as to the four flex lines on I.*

* B., section 1. By using $\tau = -2\tau$ in (7) Bianchi's form is obtained.

† Segre: *Remarques sur les transformations uniformes des courbes elliptiques en elles-mêmes*, Mathematische Annalen, vol. 27, 1886.

‡ The inflexions of Q_3 can be permuted in 216 different ways. If one is fixed as in the present case, the four flex lines on it can at the most be permuted among themselves. It is therefore to be expected that their parameters form an equianharmonic quartic invariant under a tetrahedral group.

For the irrationality $\sqrt{a_i^4}$

$$g_2 = \frac{3\tau}{4}(\tau^3 + 8), \qquad g_3 = \frac{1}{8}(\tau^6 - 20\tau^3 - 8), \qquad \Delta = -27(1-\tau^3)^3.$$

The syzygy (4) becomes

$$J : J - 1 : 1 = \tau^3(\tau^3 + 8)^3 : (\tau^6 - 20\tau^3 - 8)^2 : 64(\tau^3 - 1)^3,$$

the form of the modular equation* given by Klein. Hence τ is the tetrahedral irrationality.†

III. The elliptic norm curve of fourth order, Q_4, in three-dimensional space, S_3

9.‡ The coördinates of a point on the elliptic norm curve of fourth order, Q_4, in three-dimensional space S_3 are given by

$$x_\alpha(u) = c_\alpha \prod_{\mu=0}^{3} \sigma_{\alpha/4+1/2,\,(\mu+1/2)/4}(u\,|\,\omega_1, \omega_2) \qquad (\alpha = 0, \cdots 3),$$

where c_α is an undetermined factor and

$$\sigma_{\alpha/4,\,\mu/4}(u\,|\,\omega_1, \omega_2) \equiv e^{[(\lambda\eta_1+\mu\eta_2)/4][u-(\lambda\omega_1+\mu\omega_2)/8]}\,\sigma\left(u - \frac{\lambda\omega_1 + \mu\omega_2}{4}\,\middle|\,\omega_1, \omega_2\right).$$

To obtain a convenient expression for c_α, Klein introduces the transformed periods $\bar{\omega}_1$, $\bar{\omega}_2$, such that $\bar{\omega}_1 = \omega_1$, $\bar{\omega}_2 = \omega_2/4$. Because of the Legendre relation $\omega_1\eta_2 - \omega_2\eta_1 = 2\pi i$, the corresponding substitution for η is $\bar{\eta}_1 = 4\eta_1$, $\bar{\eta}_2 = \eta_2$. The coördinates are finally given in the form

$$x_\alpha(u\,|\,\omega_1, \omega_2) = a_\alpha\, e^{-[(\eta_1-4\eta_1)/2\omega_1]u^2+([(\alpha+2)/4]\bar{\eta}_1+\bar{\eta}/2)u}$$

$$\cdot\, e^{-([(\alpha+2)/4]\bar{\eta}_1+\bar{\eta}_2/2)([(\alpha+2)/8]\omega_1+\omega_2/16)}$$

$$\cdot\, \sigma\left(u - \frac{\alpha+2}{4}\omega_1 - \frac{\mu+\frac{1}{2}}{4}\omega_2\,\middle|\,\omega_1, \frac{\omega_2}{4}\right),$$

where

$$a_\alpha = e^{-1/2[(\pi i\alpha)/4+(5\pi i)/2]}\mathrm{X},$$

X being a function of the periods, independent of u and α. The notation indicates that $x_\alpha(u)$ depends upon the periods ω_1, ω_2, the σ functions upon the periods ω_1, $\omega_2/4$. In the coördinate system used $x_{\alpha+4}(u) = x_\alpha(u)$.

Four points u_1, u_2, u_3, u_4, of Q_4 are on a plane if

$$u_1 + u_2 + u_3 + u_4 \equiv 0 \pmod{\omega_1, \omega_2}. \tag{9}$$

* K. F., I, p. 104.

† See B., section 1, for further discussion of τ.

‡ §§ 9 and 10 do little more than give for the case $n = 4$ the general theory stated by Klein for Q_n (K. F., II, pp. 261–265).

There are 16 singular points (points of hyperosculation), which lie by fours on the faces of a tetrahedron T. These sets of fours are

$$(10) \qquad \frac{1}{2}P, \quad \frac{\omega_1}{4}+\frac{1}{2}P, \quad \frac{\omega_2}{4}+\frac{1}{2}P, \quad \frac{\omega_1+\omega_2}{4}+\frac{1}{2}P,$$

where

$$\frac{1}{2}P = 0, \quad \frac{\omega_1}{2}, \quad \frac{\omega_2}{2}, \quad \frac{\omega_1+\omega_2}{2}.$$

10. The curve Q_4 admits a collineation group G_{32} determined by the 32 substitutions

$$(11) \qquad u' = \pm u + \frac{p\omega_1 + q\omega_2}{4}^* \qquad (p, q = 0, 1, 2, 3).$$

Under G_{32}, Q_4 is invariant; and under (11), (9) is invariant. The G_{32} is defined abstractly by K_1, K_2, K_3, such that

$$K_1^2 = K_2^4 = K_3^4 = 1, \qquad K_2 K_3 = K_3 K_2,$$
$$K_1 K_2 = K_2^3 K_1, \qquad K_1 K_3 = K_3^3 K_1.$$

These generators can be taken to be

$$(12) \qquad u' = -u, \quad u' = u - \frac{\omega_1}{4}, \quad u' = u - \frac{\omega_2}{4}, \quad \text{i. e.,}$$
$$x_a(-u) = x_{4-a}(u), \qquad x_a\left(u - \frac{\omega_1}{4}\right) = x_{a+1}(u),$$
$$x_a\left(u - \frac{\omega_2}{4}\right) = \epsilon^a x_a(u),\dagger \qquad (\epsilon = e^{(2\pi i)/4}).$$

The G_{32} contains an invariant Abelian G_{16} generated by K_2 and K_3. This G_{16} contains three pairs of cyclic G_4's, each pair having a common element of period two. The three elements of period two, K_2^2, K_3^2, $K_2^2 K_3^2$, and identity constitute a four-group whose parametric expression is

$$u' = u + \tfrac{1}{2}P.$$

By adjoining K_1 four new elements of period two, which with the four-group form an Abelian G_8, are obtained. Their parametric expression is

$$u' = -u + \tfrac{1}{2}P.$$

From (10) they are evidently reflexions in vertex and opposite plane of T. Thus the unique tetrahedron T can be determined from the K's. By the transformations

$$y_0 = x_0 + x_2, \qquad y_1 = x_1 + x_3, \qquad y_2 = x_0 - x_2, \qquad y_3 = x_1 - x_3,$$

* K. F., II, pp. 237–242; and Segre, loc. cit.

† For factors of proportionality see K. F., II, p. 264.

T is made the reference tetrahedron and K_1, K_2, K_3 take the forms

$$C_1 \qquad\qquad C_2$$

$$y_0(-u) = y_0(u) \qquad y_0\left(u - \frac{\omega_1}{4}\right) = y_1(u)$$

$$y_1(-u) = y_1(u) \qquad y_1\left(u - \frac{\omega_1}{4}\right) = y_0(u)$$

$$y_2(-u) = y_2(u) \qquad y_2\left(u - \frac{\omega_1}{4}\right) = y_3(u)$$

$$y_3(-u) = -y_3(u) \qquad y_3\left(u - \frac{\omega_1}{4}\right) = -y_2(u)$$

$$C_3$$

$$y_0\left(u - \frac{\omega_2}{4}\right) = y_2(u)$$

$$y_1\left(u - \frac{\omega_2}{4}\right) = iy_3(u)$$

$$y_2\left(u - \frac{\omega_2}{4}\right) = y_0(u)$$

$$y_3\left(u - \frac{\omega_2}{4}\right) = -iy_1(u).$$

11. The pencil of quadrics on Q_4 has a unique self-polar tetrahedron whose vertices are the vertices of cones on Q_4. Since the lines joining corresponding points u, $-u$, and v, $-v$ under $u' = -u$ intersect, this tetrahedron is evidently T. Thus G_{32} leaves the pencil of quadrics unaltered and permutes the four cones under a four-group, which is the factor group of G_{32} with reference to the G_8 mentioned in § 10.

If the cone with vertex opposite $y_3 = 0$ is

$$ay_0^2 + by_1^2 + cy_2^2 = 0, \tag{13}$$

the others furnished by the four-group are

$$by_0^2 + ay_1^2 + cy_3^2 = 0, \tag{14}$$

$$cy_0^2 + ay_2^2 - by_3^2 = 0, \tag{15}$$

$$cy_1^2 - by_2^2 + ay_3^2 = 0. \tag{16}$$

These are in a pencil if

$$a^2 = b^2 + c^2 \tag{17}$$

or if

$$(18) \qquad a = \tau_1^2 + \tau_2^2, \qquad b = \tau_1^2 - \tau_2^2, \qquad c = 2\tau_1\tau_2.$$

As τ varies the family of Q_4's, each of which admits the same G_{32}, is obtained. This family lies on the surface

$$(19) \qquad y_0^4 - y_1^4 - y_2^4 + y_3^4 = 0,$$

obtained by eliminating τ.

12. Let $P_0 \cdots P_3$ be the vertices of T; $L_i = \overline{P_0 P_i}$, $L_i' = \overline{P_j P_k}$, $(i, j, k = 1, 2, 3)$, pairs of opposite edges of T. Let a cyclic G_4 (§ 10) be generated by C_2 or $u' = u - \omega_1/4$. Then C_2 is not perspective, for if it had a pencil of fixed planes, the pencil would cut the curve in sets of four conjugate co-planar points, whereas the general set u, $u - \omega_1/4$, $u - 2\omega_1/4$, $u - 3\omega_1/4$ is not coplanar. Hence C_2 has a fixed tetrahedron T_i whose vertices are on a line pair L_i, L_i' of T, since C_2^2 is a reflexion in such a line pair. The reflexions in the other line pairs leave the cyclic G_4 unaltered, whence the vertices of T_i on L_i and L_i' are harmonic with those of T. The second cyclic G_4 containing C_2^2 (§ 10) has a fixed tetrahedron T_i'. Thus the vertices of T, T_i, T_i' on L_i, L_i' form three mutually harmonic pairs on each line L_i. Since the cyclic G_4's are invariant under G_{32}, the six tetrahedra T_i, T_i' are invariant under G_{32}. The edges of the six tetrahedra T_i, T_i', exclusive of the edges common to T, form six skew quadrilaterals.

The elements of G_{32} not in G_{16} are involutory. The four of type

$$u' = -u + \tfrac{1}{2}P$$

are reflexions in vertex and opposite plane of T (vide § 10). The remaining twelve are $u' = -u + \frac{1}{4}P$ $(\frac{1}{4}P \neq \frac{1}{2}P)$. Since two pairs of corresponding points of Q_4 are not in general coplanar, these twelve collineations are reflexions in skew line pairs. If $u' = -u + \frac{1}{4}P$ is a reflexion in the line pair M_i, M_i', then, since it transforms $u' = -u + \frac{1}{2}P'$ into $u' = -u + \frac{1}{2}P + \frac{1}{2}P'$, it must interchange the vertices of T, i. e., M_i, M_i' lie across two skew edges of T say L_i and L_i'. The product of $u' = -u + \frac{1}{4}P$ and $u' = u + \frac{1}{2}P$, i. e., $u' = -u + \frac{3}{4}P$ will be another of the twelve involutions whose line pair N_i, N_i' will cross both L_i, L_i' and M_i, M_i'. Thus the twelve involutions determine six pairs of line pairs which form six skew quadrilaterals $\overline{T}_i$ with vertices on L_i, L_i'. These six quadrilaterals separate into three pairs $\overline{T}_i$, $\overline{T}_i'$, the two of a pair having their vertices on the same line pair L_i, L_i'.

The involution on M_i, M_i' leaves unaltered the line L_i and the tetrahedra T_i, T_i'. Since it cannot interchange all three pairs of vertices of T_i, T_i', T on L_i the pair must cross L_i at the vertices of T_i or T_i'. Hence: *The six skew quadrilaterals T_i, T_i' and the six skew quadrilaterals $\overline{T}_i$, $\overline{T}_i'$ have the same vertices.*

If a_i, b_i are the pairs of vertices on L_i, a_i', b_i' the pairs on L_i', and if the

quadrilaterals T_i, T_i' are formed by joining a_i to a_i', b_i to b_i', then the quadrilaterals $\overline{T}_i$, $\overline{T}_i'$ are formed by joining a_i to b_i', b_i to a_i'.

13. If an elliptic quartic degenerates, at least two of the four cones on it must coincide. From equations (13)–(16) it can be seen that this requires a or b or c to be zero. Then two pairs of cones each coincide into pairs of planes and the quartic is a skew quadrilateral. There are six such Q_4's in the family, corresponding to the values of the modulus given by

$$\alpha_\tau^6 \equiv \tau_1^5 \tau_2 - \tau_1 \tau_2^5 = 0. \tag{20}$$

On the other hand, the six quadrilaterals T_i admit G_{32} (§ 12) and occur in the family. Therefore the quadrilaterals T_i are on the quartic surface (19).

The parametric representation shows that the involution on M_i, M_i' has four fixed points on each Q_4. This line pair meets each T_i in four points. It has been shown above that for T_i and T_i' these four points are two vertices or double points. For T_j, T_j' the four points are paired on opposite edges. Thus M_i has more than four points in common with the surface (19) and therefore lies on it. But the surface has just 48 lines determined by the separation of its terms into two sets of two. Hence *the twelve skew quadrilaterals* T_i, T_i'; $\overline{T}_i$, $\overline{T}_i'$ *are the* 48 *lines of the quartic locus of the family of* Q_4*'s which admit the given* G_{32}. *The six quadrilaterals* T_i, T_i' *are the degenerate curves of the family.**

14. In order to introduce the irrationality discussed in section I, the family of Q_4's will be projected into a family of doubly-covered rational curves as was done in the case of Q_3 (section II). In that simple case the projected family lay on a single line. The projection was made by means of one of the involutions which belonged to every curve. In the present case there are two types of involution in G_{32} which are not in G_{16}. By using the type determined by a vertex and opposite plane of T, the family of Q_4's is projected upon a family of doubly-covered conics. By using the type determined by two skew lines of the quadrilateral $\overline{T}_i$ the projection is upon a family of doubly covered lines all of which lie on one line of the pair.

For the first type, if $y_3 = 0$ is the plane of the projection, the family of doubly-covered conics is from (13)

$$ay_0^2 + by_1^2 + cy_2^2 = 0. \tag{21}$$

In order to introduce a parameter t on the conic, which is covariantly related to the modulus τ of the quartic, it is noted that the reference triangle cuts out on the conic an octahedral sextic which can be identified with α_τ^6, the one which determines the degenerate curves. Then the conic can be written as follows:

* Segre, loc. cit., and *Concerning the twisted biquadratic*, by J. C. Kluyver, American Journal of Mathematics, vol. 19 (1897), p. 319 ff.

$$y_0 = \sqrt{\tau_1^2 - \tau_2^2}\,\sqrt{2\tau_1\tau_2}\,(t_1^2 + t_2^2),$$

$$(22) \qquad y_1 = i\,\sqrt{2\tau_1\tau_2}\,\sqrt{\tau_1^2 + \tau_2^2}\,(t_1^2 - t_2^2),$$

$$y_2 = i\,\sqrt{\tau_1^2 + \tau_2^2}\,\sqrt{\tau_1^2 - \tau_2^2}\,(2t_1 t_2),$$

and from (14)

$$y_3 = \pm 2\,\sqrt{2\,(t\tau)\,\alpha_\tau^3\,\alpha_\tau^3}.$$

The geometric interpretation of the radical in t can be stated thus: *If an elliptic quartic lies on a cone, the six generators on pairs of points of hyperosculation form an octahedral sextic, and the four generators on single points of hyperosculation are such that any three form the cubic polar of the remaining one as to the sextic.*

The expression for $\tau = \tau_1/\tau_2$ as an elliptic modular function can be obtained from (21) and (19) in the form

$$\tau = \frac{\tau_1}{\tau_2} = \frac{-\,y_2^2 + \sqrt{y_2^4 - y_0^4 + y_1^4}}{y_0^2 + y_1^2} = \frac{-\,y_2^2 + y_3^2}{y_0^2 + y_1^2}.$$

If this is evaluated for $u = 0$ and expressed in terms of x_i (vide § 10), the following expression is obtained

$$\tau = \frac{-\,[x_0(0) - x_2(0)]^2 + [x_1(0) - x_3(0)]^2}{[x_0(0) + x_2(0)]^2 + [x_1(0) + x_3(0)]^2}.$$

The modular equation satisfied by τ can be obtained by forming g_2, g_3, and Δ for the irrationality $\sqrt{2\,(t\tau)\,\alpha_\tau^3\,\alpha_t^3}$ as in section I by letting

$$n = 6, \qquad \alpha_t^6 = t_1^5\,t_2 - t_1\,t_2^5.$$

The equation appears in the well-known octahedral form.*

The presence of the radicals $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ in the parametric form (22) of Q_4 is due to the fact that in the plane $y_3 = 0$ there are four points of hyperosculation determined by $t = \pm\tau$, $\pm 1/\tau$, the roots of $(t\tau)\,\alpha_\tau^3\,\alpha_t^3 = 0$. One of these, $t = \tau$, has been isolated by the choice of the parameter t.

A striking form of the elliptic integral can be deduced from the type

$$u = \int \frac{(t\,dt)}{\sqrt{(t\tau)\,\alpha_\tau^3\,\alpha_t^3}}.$$

The family of Q_4's has been projected into the family of conics (21) which are intimately related to the quartic section of the family of Q_4's, made by the plane of projection. This quartic section† is

* K. F., I, p. 20.

† *Über Untersuchung und Aufstellung von Gruppe und Irrationalität regulärer Riemann'schen Flächen* and *Notiz über eine reguläre Riemann'sche Fläche vom Geschlechte drei und die zugehörige "Normalcurve" vierter Ordnung*, by W. Dyck in Mathematische Annalen, vol. 17 (1880), pp. 473 and 510. The connection is here made with the elliptic modular function, but the system of conics mentioned above is not considered. The normal forms of the elliptic integral are to be contrasted with those described in an article by Klein in the same volume.

(23) $$a_y^4 \equiv y_0^4 - y_1^4 - y_2^4 = 0$$

or parametrically

(24) $$y_0 = \sqrt{a}, \qquad y_1 = i\sqrt{b}, \qquad y_2 = i\sqrt{c}.$$

The family (21) is evidently the system of polar conics as to a_y^4, of a point y on $a_y^4 = 0$. Moreover $a_y^4 = 0$ is the envelope of this quadratic family of conics (21), which appears therefore as an isolated Steiner system of the quartic curve.

Consider the integral

(24) $$I = \int \frac{(y\, x\, xd)}{\sqrt{a_y^3\, a_x \cdot (aa'\, a'')\, a_y^3\, (a'\, a''\, yx)^3}},$$

where x and y are subject to the relations $a_y^4 = 0$, $a_y^2\, a_x^2 = 0$. The integral I depends first upon the choice of the point y on $a_y^4 = 0$, secondly upon $x_0 : x_1 : x_2$, a point which is required to be on (21), $a_y^2\, a_x^2 = 0$. That I is an elliptic integral follows from the fact that if a parameter is introduced on the rational curve the radical reduces to one of degree four. This can easily be verified. Let

$$a_1 = t_1^2 + t_2^2, \qquad b_1 = t_1^2 - t_2^2, \qquad c_1 = 2t_1\, t_2.$$

Then with y expressed as in (24) and $a_y^2\, a_x^2 = 0$ as in (22) it is found that

$$(y\, x\, dx) = -4\sqrt{a}\ \sqrt{b}\ \sqrt{c}\,(t\tau)^2\,(t\, dt).$$

Since

$$(a'\, a''\, u)^4 = 2\,(u_0^4 - u_1^4 - u_2^4),$$

$$a_y^3\, a_x = 4\,(y_0^3\, x_0 - y_1^3\, x_1 - y_2^3\, x_2),$$

(25) $$(aa'\, a'')\, a_y^3\, (a'\, a''\, u)^3 = 32\,(y_0^3\, u_0^3 + y_1^3\, u_1^3 + y_2^3\, u_2^3).$$

Instead of the line coördinates introduce the values

$$u_0 = y_1\, x_2 - y_2\, x_1 = \sqrt{a}\,(-bc_1 + b_1\, c) = \sqrt{a}\,(t\tau)\,(t_1\, \tau_1 + t_2\, \tau_2), \quad \text{etc.}$$

Then (25) becomes

(25′) $$(aa'\, a'')\, a_y^3\, (a'\, a''\, u)^3 = \frac{16}{5}\,(t\tau)^3\, \alpha_\tau^3\, \alpha_t^3,$$

and

$$a_y^3\, a_x = 8\sqrt{a}\ \sqrt{b}\ \sqrt{c}\,(t\tau)^2.$$

The integral I can now be written

$$I = -\frac{\sqrt{5}}{\sqrt[4]{abc}} \int \frac{(t\, dt)}{2\sqrt{2}\,(t\tau)\,\alpha_\tau^3\, \alpha_t^3}.$$

If $a_y^4 = 0$ is the Dyck quartic automorphic under a collineation group G_{96}, the integral

$$I = \int \frac{(y\,x\,dx)}{\sqrt{a_y^3\,a_x \cdot (aa'\,a'')\,a_y^3\,(a'\,a''\,yx)^3}}$$

is an elliptic integral of the first kind with moduli $y_0 : y_1 : y_2$, *and variables* $x_0 : x_1 : x_2$ *on the polar conic of* y *as to the quartic. This integral is unaltered if the variables and moduli are cogrediently transformed under* G_{96}.

If $a_y^4 = 0$ *is the Klein quartic invariant under the collineation group* G_{168} *the integral* I *has the same properties and covariant character.*

The last statement is given here without proof. The theorem indicates the possibility of extending the canonical form obtained for genus zero to larger values of the genus and gives the form applicable to genus three.

For the second type of projection mentioned in the beginning of this paragraph, the involution $C_1\,C_2$ or parametrically $u' = -u + \omega_1/4$ is used. A pencil of planes on one fixed line of $C_1\,C_2$ is given by

$$y_0 + y_1 + i\sqrt{\tau\lambda}\,(y_2 + y_3) = 0. \tag{26}$$

By eliminating y_0/y_3, y_1/y_3 from (26), (13), and (14) the value of y_2/y_3 for the points of intersection of the plane (26) and Q_4 is found to be

$$\frac{y_2}{y_3} = \frac{[a+b]\,[c^2 - \tau^2\lambda^4\,(a-b)^2] \mp 2\,\sqrt{\tau\lambda}c\,\sqrt{8\,(\lambda\tau)\,\alpha_\tau^3\,\alpha_\lambda^3}}{\tau^2\lambda^4\,c^2\,(a-b) - 2\tau\lambda^2\,c\,(a-b)^2 + c^2\,(a+b)}. \tag{27}$$

The irrationality in (27) which separates corresponding points of Q_4 is again in the canonical form.*

15. Since τ is an elliptic modular function and the octahedral irrationality, it admits the group Γ_{24} the principal subgroup of Γ_{48}, which is generated by

$$\mathbf{S}\quad \begin{matrix}\omega_1' = \omega_1 + \omega_2 \\ \omega_2' = \omega_2\end{matrix} \qquad \mathbf{T}\quad \begin{matrix}\omega_1' = -\omega_2 \\ \omega_2' = \omega_1,\end{matrix}$$

where the determinants of the transformations are congruent to identity (mod 4). The effect of **S** and **T** on the coördinates of a point of Q_4 is given by

$$\mathbf{S}_1 : y_0' = \sqrt{i}\,y_2, \quad y_1' = iy_1, \quad y_2' = \sqrt{i}\,y_0, \quad y_3' = iy_3$$

$$\mathbf{T}_1 : y_0' = y_0, \quad y_1' = y_2, \quad y_2' = y_1, \quad y_3' = -iy_3;\dagger$$

on the modulus τ, by

* The explicit equations of the quadrics generated by the lines joining corresponding points of Q_0 under $C_1\,C_2$ and the eleven other involutions of the same type can be derived (vide also Segre, loc. cit.). The parameters of these six quadrics as members of the pencil (13) $+ \mu$ (14) $= 0$ form an octahedral sextic.

† K. F., II, p. 299, (12) and (13). There is a misprint in (12) but the form is given correctly on p. 297. Klein gives the transformation on x's for Q_3.

$$\mathbf{S}_2 : \tau' = \frac{\tau - i}{\tau + i} \quad \text{or} \quad \tau' = \frac{\tau + i}{\tau - i},$$

$$\mathbf{T}_2 : \tau' = \frac{\tau + 1}{\tau - 1} \quad \text{or} \quad \tau' = \frac{\tau - 1}{\tau + 1}.$$

The first substitutions of $\mathbf{S}_2$ and $\mathbf{T}_2$ are sufficient to generate the octahedral group G_{24} on τ.

By the addition of G_{24} to G_{32} a $G_{32\cdot 24}$ is obtained under which the quartics of the family are at the most permuted among themselves.

If now the surface

$$y_0^4 - y_1^4 - y_2^4 + y_3^4 = 0 \tag{19}$$

is considered, it will readily be seen that (19) is unaltered by the twenty-four permutations and the four letters y_i combined with the roots of -1, of which there are three choices. Hence a $G_{24\cdot 4\cdot 4\cdot 4}$ leaves (19) unaltered. But a $G_{32\cdot 24}$ has already been accounted for. The one type of transformation not yet included is

$$C_4 \qquad y_0' = y_0, \quad y_1' = y_1, \quad y_2' = y_2, \quad y_3' = iy_3.$$

The surface (19) is invariant under C_4 but its two sets of twenty-four lines are interchanged, i. e., the axes of the twelve involutory collineations of type $C_1 C_2$ which form the skew quadrilaterals $\overline{T}_i$, $\overline{T}'_i$, are transformed into the degenerate curves forming the skew quadrilaterals T_i, T'_i and vice versa.* This indicates the existence of two families of Q_4's on the surface (19) arranged as are the generators of a quadric in that through every point of the surface passes a curve of each family. But a curve of one family cuts every curve of the other family in eight points.

IV. The elliptic norm curve of the fifth order, Q_5, in four-dimensional space, S_4

16. The reader is referred to the frequently cited article of Bianchi for the preliminary work needed in the discussion of the elliptic curve of fifth order, Q_5, in four-dimensional space S_4. In accordance with the notation of that article, the six fundamental pentahedra will be designated by P_∞, P_i $(i = 0, \cdots, 4)$. The collineations determined by the substitutions $u' = -u$, $u' = u - \omega_1/5$, $u' = u + \omega_2/5$ will be represented by C_1, C_2, C_3 respectively. These are the generators of the fifty collineations, G_{50}, under which Q_5 is invariant. The five quadrics whose common intersection is Q_5 are

$$\phi_i \qquad \tau_1 \tau_2 x_i^2 + \tau_1^2 x_{i+2} x_{i+3} - \tau_2^2 x_{i+1} x_{i+4} = 0\dagger \quad (i = 0, \cdots 4;\ x_{i+5} = x_i).$$

* See also Kluyver, loc. cit.

† B., § 13. The a of Bianchi is replaced by τ_1 / τ_2.

In order to develop the elliptic irrationality it is convenient to introduce coördinates suggested by the involution C_1. Let

$$y_0 = x_0, \quad y_1 = x_1 + x_4, \quad y_2 = x_2 + x_3,$$
$$z_1 = x_1 + x_4, \quad z_2 = -x_2 + x_3.*$$

Then C_1 has a plane π of fixed points determined by $z_1 = z_2 = 0$, in which the coördinates are $y_0 : y_1 : y_2$; and a skew line L of fixed points determined by $y_0 = y_1 = y_2 = 0$, in which the coördinates are $z_1 : z_2$. The explicit form of C_1 in the new coördinate system† is

$$C_1': \quad y_i(-u) = -y_i(u), \quad z_j(-u) = z_j(u) \quad (i = 0, 1, 2; j = 1, 2).$$

Since corresponding points in the involution are cut out by a pencil of S_3's on π, such points will have for their coördinates the same y_i's and the z_j's differing by the sign of a radical. The radical can easily be obtained.

17. The quadrics ϕ_i can be rewritten in terms of y and z.

$$\phi_0'. \quad 4\tau_1\tau_2 y_0^2 + \tau_1^2 y_2^2 - \tau_2^2 y_1^2 = (\tau_1 z_2 + \tau_2 z_1)(\tau_1 z_2 - \tau_2 z_1),$$

$$\phi_1'. \quad \tau_1\tau_2 y_1^2 + \tau_1^2 y_1 y_2 - 2\tau_2^2 y_0 y_2 = \tau_1 z_1(\tau_1 z_2 - \tau_2 z_1),$$

$$\phi_2'. \quad \tau_1\tau_2 y_2^2 + 2\tau_1^2 y_0 y_1 - \tau_2^2 y_1 y_2 = -\tau_2 z_2(\tau_1 z_2 - \tau_2 z_1),$$

$$\phi_3'. \quad 2\tau_2^2 z_2 y_0 + (\tau_1^2 z_2 + 2\tau_1\tau_2 z_1) y_1 - \tau_1^2 z_1 y_2 = 0,$$

$$\phi_4'. \quad 2\tau_1^2 z_1 y_0 + \tau_2^2 z_2 y_1 + (\tau_2^2 z_1 + 2\tau_1\tau_2 z_2) y_2 = 0.$$

Since ϕ_3' and ϕ_4' are homogeneous in y_i, they can be solved for y_i.

$$y_0 = -\rho\tau_1\tau_2[\tau_2^2 z_1^2 + 3\tau_1\tau_2 z_1 z_2 + \tau_1^2 z_2^2],$$
$$y_1 = \rho[\tau_1^4 z_1^2 + \tau_2^4 z_1 z_2 + 2\tau_1\tau_2^3 z_2^2],$$
$$y_2 = \rho[2\tau_1^3\tau_2 z_1^2 + \tau_1^4 z_1 z_2 - \tau_2^4 z_2^2],$$

where ρ is an undetermined factor of proportionality. By substituting these values in ϕ_0', ϕ_1', ϕ_2' the condition which ρ must satisfy is found to be

$$\rho^2\{z_1^3[4\tau_1^3\tau_2^6 + 3\tau_1^8\tau_2] + z_1^2 z_2[18\tau_1^4\tau_2^5 + \tau_1^9] + z_1 z_2^2[18\tau_1^5\tau_2^4 - \tau_2^9] + z_2^2[4\tau_1^6\tau_2^3 - 3\tau_1\tau_2^8]\} = \tau_1 z_2 - \tau_2 z_1. \tag{28}$$

Let $\alpha_z^{12} \equiv z_1 z_2[z_1^{10} + 11z_1^5 z_2^5 - z_2^{10}]$. Then (28) can be written

$$4\rho^2 \alpha_\tau^9 \alpha_z^3 = (\tau z). \tag{28'}$$

$$\therefore \quad \rho = \frac{(\tau z)}{2\sqrt{(\tau z)\,\alpha_\tau^9 \alpha_z^3}}.$$

* Whenever a collineation is indicated by an accented letter its form in terms of y and z is meant.

† K. F., II, p. 267 especially (13).

If this value of ρ is used, Q_5 can be written parametrically in terms of z.

$$
\begin{aligned}
y_0 &= -\tau_1\tau_2(\tau z)[\tau_2^2 z_1^2 + 3\tau_1\tau_2 z_1 z_2 + \tau_1^2 z_2^2],\\
y_1 &= (\tau z)[\tau_1^4 z_1^2 + \tau_2^4 z_1 z_2 + 2\tau_1\tau_2^3 z_2^2],\\
y_2 &= (\tau z)[2\tau_1^3\tau_2 z_1^2 + \tau_1^4 z_1 z_2 - \tau_2^4 z_2^2],\\
z_1 &= 2z_1\sqrt{(\tau z)\alpha_\tau^9\alpha_z^3},\\
z_2 &= 2z_2\sqrt{(\tau z)\alpha_\tau^9\alpha_z^3}.
\end{aligned}
\tag{29}
$$

Now consider a pencil of S_3's on π, given by

$$t_2 z_1 = t_1 z_2. \tag{30}$$

If in (29) z is replaced by t, (29) gives the coördinates of pairs of points in the involution, which by (30) project into one point on L. The notation is then exactly analogous to that used in earlier sections of this paper. But z can equally well be thought of as the variable parameter along a line. In either notation the irrationality which appears is of the type discussed in section I. The icosahedral equation can be developed in terms of τ, and τ is therefore the icosahedral irrationality.*

18. Now project Q_5 by planes on L, upon π. Since a plane in S_4 in general meets another plane in S_4 in only one point, involutory pairs on Q_5 project into single points. The locus of these points on π is the conic whose parametric expression (parameter z) is furnished by y_i of (29). The point equation of this doubly-covered conic is

$$2\tau_1^2\tau_2^2 y_0^2 - \tau_1\tau_2^3 y_1^2 + \tau_1^3\tau_2 y_2^2 + \tau_1^4 y_0 y_1 - \tau_1^2\tau_2^2 y_1 y_2 + \tau_2^4 y_0 y_2 = 0. \tag{31}$$

The line L meets Q_5 in a singular point of Q_5, whose elliptic parameter is $u = 0$, and whose parameter in (29) is $z = \tau$. The locus of the projection of this point on π is the rational sextic

$$
\begin{aligned}
y_0 &= -5z_1^3 z_2^3,\\
y_1 &= z_1^6 + 3z_1 z_2^5,\\
y_2 &= 3z_1^5 z_2 - z_2^6.
\end{aligned}
\tag{32}
$$

The conic (31) is the osculant conic of (32) at the point $z = \tau$. With every Q_5 in the family is associated a point of the sextic (32). Hence

The family of elliptic quintics in S_4, which admit a given G_{50} of collineations, is projected from the fixed line upon the fixed plane of one of the twenty-five involutory collineations of G_{50} into a family of osculant conics of a rational

* See B. also.

sextic in that plane. Through every point of the plane pass four conics of the system.

19. A consideration of the projection of Q_5 leads to a determination of the order r of the surface ψ which is the locus of the family of Q_5's. Since L is on ψ, an S_3 on L cuts ψ in L and some curve C of order $r-1$, which meets L in s points. In a given S_3, an S_2 on L cuts C in s points on L and $r-1-s$ further points. But it has been noted that through every point of π pass four conics and that to every point of a conic corresponds two points of a particular Q_5. Hence to every point of the plane corresponds eight points of the surface ψ. Therefore

$$r-1-s=8.$$

Now the S_3 on L, cutting ψ in L and C contains two directions on the surface ψ at every point of intersection of L and C, and therefore contains the tangent S_2 at such a point. But the section of S_3 by π is a line and the projection of points of contact of tangent planes is the sextic (32). Since a line cuts a sextic in six points, $s=6$.

$$\therefore \quad r=15.$$

20. In the preceding paragraph one line, L, on ψ has been mentioned. This however is but one of twenty-five such lines which are the fixed lines of the involutory collineations in G_{50}. These lines lie by fives on S_3's. Five S_3's containing such an arrangement of these lines form one of the six fundamental pentahedra.

Another set of lines on ψ are those composing the degenerate Q_5's obtained by letting τ equal the roots of

$$\alpha_\tau^{12} \equiv \tau_1\tau_2(\tau_1^{10}+11\tau_1\tau_2-\tau_2^{10})=0.$$

Each degenerate curve is a skew pentagon. The twelve pentagons can be arranged in six pairs, each pair forming edges of a fundamental pentahedron. The pairs are given by

$$\tau_1\tau_2=0$$

and

$$[\tau_1-\epsilon^\nu(\epsilon+\epsilon^4)\tau_2][\tau_1-\epsilon^\nu(\epsilon^2+\epsilon^3)\tau_2]=0$$

$$(\nu=0,\cdots 4;\ \epsilon=e^{(2\pi i)/5}).$$

21. In a study of the configurations in the plane π, the transformations arising through the modular substitutions $\boldsymbol{S}$ and $\boldsymbol{T}$ are valuable. Bianchi has given the two transformations also called $\boldsymbol{S}$ and $\boldsymbol{T}$ in terms of x. They with C_1, C_2, C_3 generate $G_{50\cdot 60}$ under which the members of the family of Q_5's in S_4 are at the most permuted. Under $G_{50\cdot 60}$, G_{50} is an invariant subgroup. The factor group is a G_{60} isomorphic with the modular substitu

tions. The subgroup of $G_{50 \cdot 60}$ leaving C_1 unaltered is a $G_{2 \cdot 60}$. It has an invariant G_2, $\{1, C_1\}$, whose factor group is isomorphic with the G_{60} already mentioned. This may be represented either by the transformations on $z_1 : z_2$ or those on $y_0 : y_1 : y_2$. The substitutions in π which generate such a G_{60} are obtained by setting $z_1 = z_2 = 0$ in $\boldsymbol{T}$ and $(C_1 \boldsymbol{S}^2)^3 \equiv \boldsymbol{S}_1'$. The explicit expressions are

	$\boldsymbol{S}_1'$	$\boldsymbol{T}'$
y_0'	$= y_0$	$= y_0 + y_1 + y_2$
y_1'	$= \epsilon^2 y_1$	$= 2y_0 + (\epsilon + \epsilon^4) y_1 + (\epsilon^2 + \epsilon^3) y_2$
y_2'	$= \epsilon^3 y_2$	$= 2y_0 + (\epsilon^2 + \epsilon^3) y_1 + (\epsilon^4 + \epsilon) y_2$
z_1'	$= -\epsilon^2 z_1$	$= (\epsilon - \epsilon^4) z_1 + (\epsilon^3 - \epsilon^2) z_2$
z_2'	$= -\epsilon^3 z_2$	$= (\epsilon^3 - \epsilon^2) z_1 + (\epsilon^4 - \epsilon) z_2$*

Let us now consider the degenerate Q_5's and their projections in π. To particularize let $\tau_1 = 0$ or $\tau_2 = 0$. Either condition applied to ϕ_i gives the five S_3's of P_∞, the reference pentahedron in the x coördinate system. These S_3's combined three at a time give the ten lines† which in terms of y_i and z_j are

	q_1'	q_2'	q_3'	q_4'	q_5'
	$y_0 = 0$	$y_0 = 0$	$y_1 = 0$	$y_1 = 0$	$y_0 = 0$
Q_5'	$y_2 = 0$	$y_1 + z_1 = 0$	$y_2 + z_2 = 0$	$y_2 - z_2 = 0$	$y_2 - z_2 = 0$
	$z_2 = 0$	$y_2 + z_2 = 0$	$y_1 - z_1 = 0$	$z_1 = 0$	$y_1 - z_1 = 0$
	q_1''	q_2''	q_3''	q_4''	q_5''
	$y_0 = 0$	$y_0 = 0$	$y_1 + z_1 = 0$	$y_2 = 0$	$y_0 = 0$
Q_5''	$y_1 = 0$	$y_1 + z_1 = 0$	$y_2 = 0$	$z_2 = 0$	$y_2 + z_2 = 0$
	$z_1 = 0$	$y_2 - z_2 = 0$	$z_2 = 0$	$y_1 - z_1 = 0$	$y_1 - z_1 = 0$

The double points of the degenerate curves are the vertices of P_∞. They project into the vertices of a triangle, Δ_∞, the reference triangle in the fixed

* It is noteworthy that the transformations $\boldsymbol{T}_1'$ and $\boldsymbol{S}_1'$ on the ternary variables y_i are contragredient to the corresponding transformations on three quadratics in the binary domain, e. g., if $A_0 = -z_1 z_2$, $A_1 = z_2^2$, $A_2 = -z_1^2$ the result of operating with $\boldsymbol{T}'$ is

$$A_0' = A_0 + A_1 + A_2, \qquad A_1' = 2A_0 + (\epsilon^2 + \epsilon^3) A_1 + (\epsilon + \epsilon^4) A_2,$$
$$A_2' = 2A_0 + (\epsilon + \epsilon^4) A_1 + (\epsilon^2 + \epsilon^3) A_2.$$

The result of operating with $\boldsymbol{S}_1'$ is

$$A_0' = A_0, \qquad A_1' = \epsilon A_1, \qquad A_2' = \epsilon^4 A_2.$$

See Klein *Vorlesungen über das Ikosaeder*, pp. 211 ff.

† The lines are ordered as is indicated by the subscripts. The arrangement given is the unique one in which the ten lines form two skew pentagons so that all the lines are included, none are repeated. It was determined by a consideration of corresponding points and lines under C_1'.

plane π of C_1'. The vertices e_1 and e_2 of Δ_∞ are doubly covered, e_0 singly covered. The lines q_1' and q_1'' are unaltered by C_1', while the remaining lines of each degenerate curve are interchanged by pairs. The lines q_1' and q_1'' cut π in the points e_1 and e_2 which are therefore the projections of q_1' and q_1'' respectively. The table shows the entire projection of Q_5' and Q_5''.

q_1'	projects	into	e_1,	q_1''	projects	into	e_2,
q_2'	"	"	ϵ_0,	q_2''	"	"	ϵ_0,
q_3'	"	"	ϵ_1,	q_3''	"	"	ϵ_2,
q_4'	"	"	ϵ_1,	q_4''	"	"	ϵ_2,
q_5'	"	"	ϵ_0,	q_5''	"	"	ϵ_0.

Now the triangle Δ_∞ is also obtained by taking the intersections of the S_3's of P_∞ with π. Since P_i can be obtained from P_∞ by operating with $\boldsymbol{TS}^i$, the remaining five triangles, Δ_i, in π can be found by operating on Δ_∞ with the substitutions in the plane which correspond to $\boldsymbol{TS}^i$ of S_4.

It has been seen that $z = \tau$ gives the tangent plane at the point where the fixed line L of C_1' cuts Q_5 (vide § 18). For Q_5' and Q_5'', L cuts q_1' and q_1''. But $\tau_2 = z_2 = 0$ gives in π the point e_1 and $\tau_1 = z_1 = 0$ gives in π the point e_2 (vide (29)). Hence the modulus attached to Q_5' is $\tau_2 = 0$, to Q_5'' is $\tau_1 = 0$.

In the triangle Δ_∞ the side ϵ_0 is isolated. This isolated side is a double flex tangent of the sextic (32). The parameters of the two flexes are $z_1 = 0$, $z_2 = 0$. The parameters of all the flexes of (32) are given by the icosahedral form α_s^{12}. The remaining double flex tangents are derived from $y_0 = 0$ by operating with $\boldsymbol{T}'\boldsymbol{S}''^{\nu}$ $(\nu = 0, \cdots, 4)$. The result is

$$y_0 + \epsilon^{2\nu} y_1 + \epsilon^{3\nu} y_2 = 0.$$

The six double flex tangents are isomorphic with the six lines of Klein's icosahedral configuration in the plane* (vide Klein, *Vorlesungen über das Ikosaeder*, pp. 211 ff). The invariant conic of G_{60} in the plane is the conic on the flexes

$$y_0^2 + y_1 y_2 = 0.$$

It cuts out on the sextic the icosahedral points.

In the plane π lie three double points of the involution C_1'. They can be obtained by setting $z_1 = z_2 = 0$ in ϕ_0', ϕ_1', ϕ_2'. If from the forms ϕ_i' thus modified, τ_1^2, $\tau_1\tau_2$, τ_2^2 are eliminated, a sextic is obtained as the locus of the three fixed points of C_1', lying in π, for the whole family of Q_5's. Its equation is

* The rest of the configuration is not particularly interesting in connection with Q_5 so it is not given. A more complete discussion of the sextic (32) is given by R. M. Winger, *Self-projective rational sextics*, American Journal of Mathematics, vol. 38, January, 1916.

$$y_0(y_1^5 + y_2^5) + 2y_0^2 y_1^2 y_2^2 - 8y_0^4 y_1 y_2 - y_1^3 y_2^3 = 0. \tag{33}$$

In § 18 the point $z = \tau$ on the doubly-covered conic (31) was accounted for. Its locus was (32), the locus of the fixed point of C_1' found on L. The fixed points of C_1' found on π are given by $\alpha_\tau^2 \alpha_z^3 = 0$, and their locus has now been found to be (33).

22. Return for a moment to a consideration of the projection of Q_5 by a pencil of S_3's on π,

$$t_2 z_1 = t_1 z_2 \tag{30}$$

to determine what happens when t is a root of $\alpha_t^{12} = 0$.

The fixed planes of the involutory collineations of G_{50} in S_4 lie by fives on the five vertices of P_∞. There are as many such arrangements as there are pentahedra. Hence there are thirty points on each of which is a fixed plane. In every plane there are six points. *The twenty-four points other than the six in the plane* $z_1 = 0, z_2 = 0$, *lie by pairs on* S_3*'s in the pencil* (30) *where* t *assumes the values of the roots of* $\alpha_t^{12} = 0$.

ON THE NOTION OF SUMMABILITY FOR THE LIMIT OF A FUNCTION OF A CONTINUOUS VARIABLE*

BY

L. L. SILVERMAN

1. INTRODUCTION†

A sequence of numbers $u(m)$ may be regarded as a function of a discrete variable m. To generalize the notion of the limit of this sequence, a new sequence $v(m)$ is defined by means of the transformation

$$v(m) = \sum_{n=1}^{m} k(m, n)\, u(n),$$

where $k(m, n)$ is a function of two discrete variables. This transformation is *regular* if the existence of

$$\lim_{m=\infty} u(m)$$

implies the existence of

$$\lim_{m=\infty} v(m)$$

and the equality of the two limits. Examples of regular transformations are those of Cesàro and Hölder, of which the following are simple cases:

$$(1) \qquad v_1'(m) = v_1''(m) = \frac{1}{m}\sum_{n=1}^{m} u(n), \qquad v_2'(m) = \frac{\sum_{n=1}^{m}\sum_{p=1}^{n} u(p)}{\sum_{n=1}^{m} n},$$

$$v_2''(m) = \frac{1}{m}\sum_{n=1}^{m} v_1''(n),$$

where the superscripts refer respectively to the definitions of Cesàro and Hölder and the subscripts to the order of summability. A necessary and sufficient condition that a transformation be regular is:‡

$$(2) \qquad (a)\ \lim_{m=\infty}\sum_{n=1}^{m} k(m, n) = 1, \qquad (b)\ \lim_{m=\infty} k(m, n) = 0,$$

$$(c)\ \sum_{n=1}^{m} |k(m, n)| < A.$$

* Presented to the Society, January 1, 1915.

† I wish to express my indebtedness to Professor W. A. Hurwitz for a number of suggestions.

‡ Toeplitz, Prace matematyczno-fizyczno, vol. 22 (1911), p. 113.

It is natural to replace the discrete variable m by the continuous variable* x, $u(m)$ by $u(x)$, the auxiliary function $k(m, n)$ by $k(x, y)$, a function of two continuous variables, the sign of summation by an integral sign, and then to study the transformation

$$v(x) = \int_a^x k(x, y)\, u(y)\, dy, \tag{3}$$

where $u(x)$ is bounded and integrable, $a \leqq x \leqq x_1$, and $k(x, y)$ is integrable in y for each x, $a \leqq y \leqq x$. The function $k(x, y)$ is the *kernel* of the transformation. The transformation is *regular* if $\lim_{x=\infty} u(x)$ implies the existence of $\lim_{x=\infty} v(x)$ and the equality of the two limits. Corresponding to the examples (1) we have, taking for convenience $a = 0$, the transformations:†

$$v_1'(x) = v_1''(x) = \frac{1}{x}\int_0^x u(y)\, dy, \qquad v_2'(x) = \frac{\int_0^x \int_0^y u(s)\, ds\, dy}{\int_0^x y\, dy}, \tag{4}$$

$$v_2''(x) = \frac{1}{x}\int_0^x v_1''(y)\, dy.$$

Changing the order of integration in the numerator of the expression for $v_2'(x)$ and integrating once, we have

$$v_2'(x) = \frac{2}{x^2}\int_0^x (x - y)\, u(y)\, dy;$$

so that the kernels for the transformations of Cesàro‡ of orders 1 and 2 are respectively

$$k_1'(x, y) = \frac{1}{x}, \qquad k_2'(x, y) = \frac{2(x - y)}{x^2}. \tag{5}$$

Following the case of series we are led to inquire whether conditions similar to (2) constitute a necessary and sufficient condition for the regularity of the transformation (3). In answer to this question we have the theorems which follow in the next section.

* The first to carry out this idea to extend Cesàro's and Hölder's definitions of summability of series to the case of integrals was Du Bois-Reymond, Journal für Mathematik, vol. 100 (1887), p. 354.

† See Landau, *Die Identität des Cesàroschen und Hölderschen Grenzwertes für Integrale*, Sächsische Berichte, vol. 65 (1913), p. 131.

‡ That is, the transformations in the case of a continuous variable corresponding to those of Cesàro for the case of sequences.

2. Conditions for regularity

THEOREM I. *Let $k(x, y)$ be defined, $a < x$, $a \leqq y \leqq x$, and integrable in y for each x; then a sufficient condition that $k(x, y)$ correspond to a regular transformation is:*

$$(6)\qquad (a)\ \lim_{x=\infty} \int_a^x k(x, y)\, dy = 1,$$

$$(b)\ \lim_{x=\infty} k(x, y) = 0 \text{ uniformly in } y, \qquad a \leqq y \leqq q,$$

$$(c)\ \int_a^x |k(x, y)|\, dy < A, \qquad a < x,$$

where q is an arbitrary constant and A is a positive constant.

We wish to prove that for any bounded and integrable function $u(x)$ the existence of $\lim_{x=\infty} u(x)$ implies the existence of

$$\lim_{x=\infty} \int_a^x k(x, s)\, u(s)\, ds$$

and the equality of the two limits.

Let $\lim_{x=\infty} u(x) = l$; then by (a) we may write,

$$l = l \int_a^x k(x, s)\, ds + \eta(x),$$

where $\lim_{x=\infty} \eta(x) = 0$; so that

$$\int_a^x k(x, s)\, u(s)\, ds - l = \int_a^x k(x, s)[u(s) - l]\, ds - \eta(x).$$

Choosing p so large that $|u(x) - l| < \epsilon$, $x \geqq p$, then holding p fixed and denoting by M a number greater than $|u(x) - l|$ in the interval $a \leqq x \leqq p$, we have for $x > p$,

$$\left| \int_a^x k(x, s)\, u(s)\, ds - l \right| \leqq \int_a^p |k(x, s)|\, |u(s) - l|\, ds + \int_p^x |k(x, s)|\, |u(s) - l|\, ds + |\eta(x)|$$

$$\leqq M \int_a^p |k(x, s)|\, ds + \epsilon \int_p^x |k(x, s)|\, ds + |\eta(x)|.$$

Since by (c)

$$\int_p^x |k(x, s)|\, ds \leqq \int_a^x |k(x, s)|\, ds < A,$$

and by (b)

$$\int_a^p |k(x, s)|\, ds < \frac{\epsilon}{M(p - a)} \int_a^p ds = \frac{\epsilon}{M}, \qquad x > X,$$

it follows that, if X' is the greater of the numbers p and X,

$$\left|\int_a^x k(x,s)u(s)\,ds - l\right| < \epsilon + \epsilon A + |\eta(x)|, \qquad x > X',$$

so that

$$\limsup_{x=\infty}\left|\int_a^x k(x,s)u(s)\,ds - l\right| \leqq \epsilon + \epsilon A.$$

Since the superior limit is less than an arbitrarily small positive constant, the actual limit must be zero. Therefore,

$$\lim_{x=\infty}\int_a^x k(x,s)u(s)\,ds = l.$$

THEOREM II. *Let $k(x,y)$ be defined, $a < x$, $a \leqq y \leqq x$, and continuous in y uniformly with respect to x, $x > h > a$, and let the zeroes of $k(x,y)$ for each x consist of a set of segments* and of a set† of points of measure zero; then a necessary and sufficient condition that $k(x,y)$ correspond to a regular transformation is that $k(x,y)$ satisfy* (6).

That these conditions are sufficient has been proved in the preceding theorem; we proceed to show that they are necessary. We accordingly assume that whenever

$$\lim_{x=\infty} u(x) = l,$$

then

$$\lim_{x=\infty}\int_a^x k(x,s)u(s)\,ds = l,$$

and we prove that each of the conditions (6) follows as a necessary consequence.

(*a*) Consider the special case in which $u(x) = 1$, $a \leqq x$, so that

$$\lim_{x=\infty} u(x) = 1;$$

then

$$\lim_{x=\infty}\int_a^x k(x,s)\,ds = 1.$$

(*b*) Consider the special case in which

$$u(x) = 1, \qquad a \leqq x \leqq y,$$
$$= 0, \qquad y < x;$$

so that $\lim_{x=\infty} u(x) = 0$.

Letting

$$\int_a^y k(x,s)\,ds = g(x,y),$$

* The set of segments is necessarily countable. See Hobson, *Theory of Functions of a Real Variable*, p. 82.

† Either of the sets may be null.

we have

$$\int_a^x k(x,s)u(s)\,ds = \int_a^y k(x,s)\,ds = g(x,y);$$

so that $\lim_{x=\infty} g(x,y) = 0$ for each y, and hence

$$\lim_{x=\infty} \frac{g(x,y+r)-g(x,y)}{r} = 0.$$

Since

$$\lim_{r=0} \frac{g(x,y+r)-g(x,y)}{r} = \frac{\partial g(x,y)}{\partial y} = k(x,y), \tag{7}$$

it will follow that

$$\lim_{x=\infty} k(x,y) = \lim_{x=\infty}\lim_{r=0} \frac{g(x,y+r)-g(x,y)}{r}$$

$$= \lim_{r=0}\lim_{x=\infty} \frac{g(x,y+r)-g(x,y)}{r} = 0,$$

if we show that for each constant y the limit in (7) is approached uniformly for all $x \geqq h$. To show this, we write

$$\frac{g(x,y+r)-g(x,y)}{r} = \frac{1}{r}\int_y^{y+r} k(x,s)\,ds = k(x,\bar{y}_x),$$

where $\bar{y}_x$ lies between y and $y+r$.

From the assumption regarding the uniform continuity of $k(x,y)$ when $y \geqq a$, we have

$$|k(x,y_1)-k(x,y)| < \epsilon, \qquad |y_1-y| < \delta, \qquad x \geqq h,$$

where ϵ is a given arbitrarily small positive constant and δ depends only upon ϵ and h. Thus for all $x \geqq h$ and for $|r| < \delta$, we have, since

$$|y-\bar{y}_x| < |r| < \delta, \qquad |k(x,\bar{y}_x)-k(x,y)| < \epsilon.$$

The limit in (7) is accordingly uniform in x. It follows that the limit of $k(x,y)$ is zero for each y.

We shall now show that

$$\lim_{x=\infty} k(x,y) = 0 \text{ uniformly in } y, \qquad a \leqq y \leqq q,$$

where q is arbitrary. Suppose this limit exists non-uniformly in y in some finite interval, $p_1 \leqq y \leqq q_1$. Then the limit must exist non-uniformly in one of the half-intervals. Subdividing the half-intervals and proceeding in the usual way, we define a value ξ of y such that in every interval about ξ the limit of $k(x,y)$ exists non-uniformly in y. We shall now obtain a contradiction by showing that in at least one interval about ξ, the limit of $k(x,y)$ exists uniformly in y.

Since $k(x, y)$ is continuous in y uniformly with respect to x, we have

$$|k(x, \xi) - k(x, y)| < \frac{\epsilon}{2}, \qquad |\xi - y| < \delta, \qquad x \geqq h,$$

and since the limit of $k(x, y)$ exists for each y, including $y = \xi$, we have

$$|k(x, \xi)| < \frac{\epsilon}{2}, \qquad x > X.$$

Adding these inequalities, we obtain

$$|k(x, y)| < \epsilon, \qquad x > X', \qquad |\xi - y| < \delta,$$

where X' is the greater of the numbers h and X. We have thus proved that the limit of $k(x, y)$ can exist non-uniformly in no finite interval; hence

$$\lim_{x=\infty} k(x, y) = 0 \text{ uniformly in } y, \qquad a \leqq y \leqq q.$$

(*c*) We are now to prove that there exists a constant A such that

$$\phi(x) = \int_a^x |k(x, s)|\, ds < A, \qquad a < x.$$

Let us assume this assertion untrue; then for some x_1, $\phi(x_1) > 10^2$. Considering the special case in which*

$$u(y) = \tfrac{1}{10} \operatorname{sgn} k(x_1, y), \qquad a \leqq y \leqq x_1,$$

so that

$$u(y)\, k(x_1, y) = \tfrac{1}{10}\, |k(x_1, y)|,$$

and letting

$$g(x) = \int_a^x k(x, s)\, u(s)\, ds,$$

we have

$$g(x_1) = \frac{1}{10} \int_a^{x_1} |k(x_1, s)|\, ds = \tfrac{1}{10}\, \phi(x_1) > 10.$$

Furthermore, from the uniform approach to zero of $k(x, y)$, proved in (*b*), it follows that

$$\lim_{x=\infty} \int_a^{x_1} |k(x, s)|\, ds = 0,$$

so that

$$\int_a^{x_1} |k(x, s)|\, ds < 1, \qquad x > X_1 > x_1. \tag{8}$$

Now choose $x_2 > X_1$ so large that

$$\phi(x_2) > 10^4 + 10 + 1. \tag{9}$$

* The function sgn z is defined, as usual, to have the value $|z|/z$ when $z \neq 0$, and the value 0 when $z = 0$.

The function $u(y)$ has been defined for the interval $a \leqq y \leqq x_1$; let us now extend the domain of definition to the interval $x_1 < y \leqq x_2$ as follows:

$$u(y) = \frac{1}{10^2} \operatorname{sgn} k(x_2, y), \qquad x_1 < y \leqq x_2.$$

Then

$$\begin{aligned} g(x_2) &= \int_a^{x_2} k(x_2, s)\, u(s)\, ds \\ &= \int_a^{x_1} k(x_2, s)\, u(s)\, ds + \int_{x_1}^{x_2} k(x_2, s)\, u(s)\, ds \\ &= \int_a^{x_1} k(x_2, s)\, u(s)\, ds + \frac{1}{10^2} \int_{x_1}^{x_2} |k(x_2, s)|\, ds \\ &= \frac{1}{10^2} \int_a^{x_2} |k(x_2, s)|\, ds - \int_a^{x_2} \left[\frac{1}{10^2} |k(x_2, s)| - k(x_2, s)\, u(s) \right] ds. \end{aligned}$$

Thus

$$\begin{aligned} |g(x_2)| &\geqq \frac{1}{10^2} \phi(x_2) - \left| \int_a^{x_1} \left[\frac{1}{10^2} |k(x_2, s)| - k(x_2, s)\, u(s) \right] ds \right| \\ &\geqq \frac{1}{10^2} \phi(x_2) - \int_a^{x_1} \left[\frac{1}{10^2} |k(x_2, s)| + |k(x_2, s)|\, |u(s)| \right] ds \\ &= \frac{1}{10^2} \phi(x_2) - \int_a^{x_1} \left(\frac{1}{10^2} + \frac{1}{10} \right) |k(x_2, s)|\, ds. \end{aligned}$$

From (8) and (9) it follows that

$$g(x_2) \geqq \frac{10^4 + 10 + 1}{10^2} - \frac{1}{10^2} - \frac{1}{10} = 10^2.$$

Continuing in this way, we can define a sequence of numbers x_n such that

$$\lim_{n=\infty} x_n = \infty, \qquad \lim_{n=\infty} |g(x_n)| = \infty;$$

therefore it is not true that the limit of $g(x)$ is zero.

On the other hand, from the hypothesis as to the roots of $k(x, y)$, $u(y)$ is integrable,* and from its definition $u(y)$ is bounded; hence, since

$$\lim_{x=\infty} u(x) = 0,$$

it follows from the fundamental property of regular transformations (§ 1) that

* The end-points of the set of intervals mentioned in the theorem form a countable set (see first footnote, p. 287); the measure of the set of all the points of discontinuity of $u(y)$ is accordingly zero. Hence $u(y)$ is integrable. See Hobson, loc. cit., p. 342.

$$\lim_{x=\infty} g(x) = \lim_{x=\infty} \int_a^x k(x, s)\, u(s) ds = 0.$$

The contradiction proves that

$$\int_a^x |k(x, s)| ds < A, \qquad a < x.$$

3. The case of infinite kernel

It will be seen that the kernels given in (5) of the transformations of Cesàro satisfy the conditions of the preceding theorem. To see whether also the kernel of the transformation* of Hölder of order 2 satisfies those conditions, let us find the expression for $k_2''(x, y)$. We have† from (4), for $x > 0$,

$$\begin{aligned} v_2''(x) &= \frac{1}{x}\int_0^x v_1''(y)\, dy = \frac{1}{x}\int_0^x \int_0^y \frac{1}{y} u(s)\, ds\, dy \\ &= \lim_{\epsilon=0} \frac{1}{x}\int_\epsilon^x \int_0^y \frac{1}{y} u(s)\, ds\, dy \\ &= \lim_{\epsilon=0} \frac{1}{x}\left[\log y \int_0^y u(s)\, ds\right]_\epsilon^x - \lim_{\epsilon=0} \frac{1}{x}\int_\epsilon^x u(y) \log y\, dy \\ &= \frac{1}{x}\int_0^x \log\left(\frac{x}{y}\right) u(y)\, dy, \end{aligned}$$

since

$$\left|\log \epsilon \int_0^\epsilon u(s)\, ds\right| < M\epsilon |\log \epsilon|,$$

where M, a constant independent of ϵ, is an upper bound of $u(x)$. Thus

$$k_2''(x, y) = \frac{1}{x} \log \frac{x}{y}.$$

This kernel does not satisfy the conditions of the preceding theorem, since it fails to be continuous for $y = 0$. We are thus led to study the transformation

$$v(x) = \int_a^x k(x, s)\, u(s)\, ds,$$

where $k(x, y)$ may become infinite for $x = a$ while the integral converges;‡ to define as *regular* any such transformation for which the existence of

$$\lim_{x=\infty} u(x)$$

* The kernel of Hölder's transformation of order 1 is identical with that of Cesàro of order 1.

† It is here assumed that $u(x)$ is continuous; everywhere else in this paper $u(x)$ is assumed to be merely bounded and integrable.

‡ As in § 1, $u(x)$ is assumed to be bounded and integrable, $a \leqq x \leqq x_1$.

implies the existence of

$$\lim_{x=\infty} \int_a^x k(x, s)\, u(s)\, ds$$

and the equality of the two limits; and to inquire whether the results of the preceding section can be extended to this class of transformations. It will be observed that the third condition of (6) will now necessitate the further assumption that

$$\int_a^x |k(x, y)|\, dy$$

converge. With the additional assumption that this integral converges uniformly in x, a theorem similar to Theorem I of the preceding section can be proved for the more general class of transformations under consideration.

THEOREM III. *Let $k(x, y)$ be defined, $a < y \leqq x$, and integrable in y for each x; then a sufficient condition that $k(x, y)$ correspond to a regular transformation is:*

$$\int_a^b |k(x, y)|\, dy \text{ converges}, \qquad \lim_{x=\infty} \int_a^b |k(x, y)|\, dy = 0,$$

$$\lim_{x=\infty} \int_a^x k(x, y)\, dy = 1, \qquad \int_a^x |k(x, y)|\, dy < A, \qquad a < x, \tag{10}$$

where b is an arbitrary constant greater than a, and A is a positive constant.

In order to prove this theorem it is necessary to bear in mind that the convergence of the integral*

$$\int_a^x |k(x, y)|\, dy$$

implies the convergence of the integral

$$\int_a^x |k(x, y)|\, |u(y)|\, dy,$$

since $u(y)$ is bounded. We may now follow the successive steps of the proof in Theorem I, observing that every integral involved is necessarily convergent.

We shall now state a necessary and sufficient condition that a kernel $k(x, y)$, for the more general class of transformations considered in this section, correspond to a regular transformation. In this theorem, however, a far greater restriction is put on the kernel than in the preceding ones; we assume, in fact,

* That is, the existence of the limit, as h approaches zero, of the integral

$$\int_{a+h}^x |k(x, y)|\, dy.$$

that the kernel is positive. It should be pointed out that the kernels corresponding to the transformations of Cesàro and Hölder satisfy this condition.

THEOREM IV. *Let $k(x, y)$ be positive or zero, $a < y \leqq x$, and integrable in y for each x; then a necessary and sufficient condition that $k(x, y)$ correspond to a regular transformation is*

$$\int_a^b k(x, y)\,dy \text{ converges;} \qquad \lim_{x=\infty} \int_a^b k(x, y)\,dy = 0; \tag{11}$$

$$\lim_{x=\infty} \int_a^x k(x, y)\,dy = 1,$$

where b is an arbitrary constant greater than a.

That the condition is sufficient follows immediately from Theorem III, if we make use of the hypothesis that $k(x, y)$ is positive; we accordingly proceed to the proof of the necessity of conditions (11). The hypothesis is that the convergence of $u(x)$ to the value l implies, for every bounded and integrable function $u(x)$, the two consequences,

$$\int_a^x k(x, s)\,u(s)\,ds$$

converges, and

$$\lim_{x=\infty} \int_a^x k(x, s)\,u(s)\,ds = l.$$

To prove the first two conditions (11), consider the special case in which

$$u(y) = 1, \qquad y \leqq b,$$
$$= 0, \qquad y > b,$$

so that $\lim_{x=\infty} u(x) = 0$. Hence, since

$$\int_a^x k(x, s)\,u(s)\,ds = \int_a^b k(x, s)\,ds,$$

the integral on the right converges; which proves the first of conditions (11). Furthermore

$$\lim_{x=\infty} \int_a^x k(x, s)\,u(s)\,ds = \lim_{x=\infty} \int_a^b k(x, s)\,ds = 0;$$

which proves the second of conditions (11). The third condition is proved as in Theorem II.

4. CONCLUSION

Three things may be said concerning the generalization for the limit of a function of a continuous variable as that variable becomes infinite. In the first place, improper integrals are included as a special case; for if we have

under consideration

$$\int_a^\infty \phi(x)\,dx,$$

we may write

$$u(x) = \int_a^x \phi(y)\,dy$$

and inquire as to the existence of

$$\lim_{x=\infty} u(x).$$

Secondly, we need not restrict ourselves to the case where the independent variable approaches infinity; if we wish to study $\lim_{x=a} u(x)$, we may make the substitution

$$x = a + \frac{1}{t}$$

and study

$$\lim_{t=\infty} u\left(a + \frac{1}{t}\right).$$

In particular we may generalize the notion of continuity. Finally, we wish to call attention to the fact that the class of transformations considered does not include the identity. We shall therefore study transformations of the form

$$v(x) = \alpha u(x) + (1 - \alpha)\int_a^x k(x, s)\,u(s)\,ds,$$

which gives for $\alpha = 0$ the transformations already considered, and for $\alpha = 1$ the identity. It is clear that the conditions for regularity obtained in the preceding section apply to this case also; in fact, the statements and proofs of Theorems I–IV may be repeated word for word for this more general class of transformations.

In another paper we shall discuss the consistency and equivalence of these transformations.

CORNELL UNIVERSITY,
August, 1915

ON THE FACTORIZATION OF CREMONA PLANE TRANSFORMATIONS*

BY

JAMES W. ALEXANDER

INTRODUCTION

1. Quite a number of results in algebraic geometry have been made to depend on Noether's fundamental theorem that a Cremona transformation of the plane can always be resolved into a product of quadratic ones. It is therefore desirable that this theorem be proved in as simple and convincing a manner as possible. Now, as was pointed out by Segre,† the proofs by Noether, Clifford, and others are incomplete and even break down for certain examples that can be effectively set up. There remains a very interesting proof by Castelnuovo‡ which is especially designed to meet Segre's objections and which, along with a footnote to the same by Segre, suggested the method exposed in Part II below. The present proof, however, besides being considerably simpler and shorter, appears to be free from certain objections that might perhaps be raised to Castelnuovo's method. These objections, which are analogous to those directed against the older proofs, will be presented in Part I.

I

2. Let N be the homaloidal net of a plane Cremona transformation and let $O_0, O_1, \cdots, O_k$ be the base points of N arranged in order of decreasing multiplicities. Castelnuovo shows that a Jonquières transformation§ of suitable order ν having its base point of order $\nu - 1$ at O_0 and its $2\nu - 2$ simple base points at $O_1, O_2, \cdots, O_{2\nu-2}$ will, if it exists, reduce the net N to one made up of curves of lower degree. Hence, if it can be shown that the transformation always exists, Noether's theorem will follow at once from the fact that every Jonquières transformation can be factored into quadratic ones.‖

* Presented to the Society, December 28, 1915.

† Segre, Atti della R. Accademia delle Scienze di Torino, vol. 36 (1901), p. 645.

‡ Castelnuovo, ibid., p. 861.

§ Defined below.

‖ See the proofs by Castelnuovo and Segre in the paper by the former already quoted. The proof in Part II does not make use of Jonquières transformations.

Now, to prove the existence of the necessary Jonquières transformation, it must be shown (a) that there exist ∞^2 curves of order ν satisfying the given conditions at the base points, and (b) that these curves are not all reducible. Under these conditions, the curves will make up the homaloidal net of the desired transformation. Castelnuovo appears to assume (a) tacitly in his proof of (b),* making an assumption which might give rise to doubts when some of the base points O_i are of the infinitely neighboring kind.† When the base points $O_0, O_1, \cdots, O_{2\nu-2}$ are all distinct, there can be no question whatsoever. The conditions imposed upon the curves of order ν cannot exceed $\nu(\nu-1)/2 + 2\nu - 2 = (\nu^2+3\nu-4)/2$, while there are $\infty^{\nu(\nu+3)/2}$ curves of this order. Hence, there exist at least ∞^2 curves satisfying the conditions. But suppose the points $O_{2\nu-2}$ and $O_{2\nu-3}$ could both be infinitely neighboring to O_1 in the neighborhood of the first order.‡ Then in order to pass through $O_{2\nu-2}$ and $O_{2\nu-3}$, a curve would necessarily have a double point at O_1, and the condition of passing through these three points would be expressed by 5 equations among the coefficients instead of by 3 as in the general case. We would then no longer expect to find ∞^2 curves satisfying all the conditions. A number of other cases are conceivable which, if they could actually arise, would lead to difficulties; we cite only two:

(α) Could not the sum of the multiplicities of those of the points $O_1, O_2, \cdots, O_{2\nu-2}$ which were infinitely neighboring of the first order to O_0 be greater than $\nu - 1$, imposing upon the curves of degree ν the condition of having at O_0 a multiple point of order greater than $\nu - 1$?

(β) When we impose on the curves of degree ν the condition of passing in the way assigned through $O_0, O_1, \cdots, O_{2\nu-3}$ might they not all in consequence be tangent at O_1 to a fixed line t. If this were possible and if at the same time $O_{2\nu-2}$ were infinitely neighboring to O_1 along a line other than t, the curves could not pass through $O_{2\nu-2}$ without having a double point at O_1.

3. The objections that were raised in the last section will perhaps not seem over-critical if it is recalled that only the impossibility of making an assumption like (a) prevents one from reducing the order of a net by means of a single quadratic transformation. It is well known that the sum of the multiplicities of the three points O_0, O_1, and O_2 is greater than the order n of the net N,§ so that if these points can be taken as the base points of a quadratic transformation, they will transform the net N into one of lower order. Further-

* Castelnuovo, "... ammesso che ciascuna delle ∞^2 curve C ... si spezzi." Loc. cit., p. 866, l. 5.

† An explanation of this terminology, which was introduced by Noether, will be found in Severi, *Lezioni di Geometria Algebrica*, Chap. II.

‡ Base points in the neighborhood of the first order are those which can be put in evidence by a *single* quadratic transformation having a fundamental point at O_1.

§ Cf. footnote on p. 298 below.

more, if there exist ∞^2 curves of the second degree passing through the points in question it is easily seen that they are not all reducible. Because if they were, their variable portion would be a general line of the plane and their fixed portion a line through O_0, O_1, and O_2, which is impossible since no line can meet a curve of the net in more than n points. It is however false to assume that there always exist ∞^2 conics satisfying the given conditions.

II

4. Noether's theorem is equivalent to the theorem that a homaloidal net can always be carried into a net of straight lines by a series of quadratic transformations. Let N be a homaloidal net of degree n and let

$$\alpha_0 \geqq \alpha_1 \geqq \cdots \geqq \alpha_k \geqq 1 \tag{1}$$

be the multiplicities of its base points

$$O_0, O_1, \cdots O_k \tag{2}$$

respectively, where (2) includes all the base points, ordinary and infinitely neighboring. We then have the two well-known relations

$$\sum_{i=0}^{k} \alpha_i^2 = (n-1)(n+1), \tag{3}$$

and

$$\sum_{i=0}^{k} \alpha_i(\alpha_i - 1) = (n-1)(n-2), \tag{4}$$

the first expressing the condition that all but one of the intersections of two curves of the net occur at the base points, the second that the curves of the net are of genus zero. Multiplying (3) and (4) by $j-1$ and j respectively, and subtracting the first from the second, we have

$$\sum_{i=0}^{k} \alpha_i(\alpha_i - j) = (n-1)(n-3j+1); \tag{5}$$

and since, when we put $j = \alpha_0$, all the terms of the sum are zero or negative, we have

$$0 \geqq (n-1)(n-3\alpha_0+1), \tag{6}$$

from which we conclude that

$$\alpha_0 \geqq \frac{n+1}{3} > \frac{n}{3}. \tag{7}$$

5. From now on, we shall put

$$j = \frac{n-\alpha_0}{2} \geqq \frac{1}{2}, \quad \text{or} \quad n = \alpha_0 + 2j, \tag{8}$$

and shall denote by h the number of base points other than O_0 of multiplicities greater than j. The multiplicities of these points are limited by the inequalities

$$2j \geqq \alpha_1 \geqq \alpha_2 \cdots \geqq \alpha_h > j \tag{9}$$

because the line $O_0 O_1$ can only meet a curve of the net N in $2j$ points aside from O_0. Moreover, their number is always greater than 1 except for a net of lines, as the following argument will show.

From the right-hand member of (5), we drop off the positive quantity $3j - 1$ and from the other, all the terms of the summation after the one in α_h, that is, all the terms which are negative or zero. We then have

$$\sum_{i=0}^{h} \alpha_i(\alpha_i - j) > n(n - 3j) = n(\alpha_0 - j);$$

or, on transposing the first term of the sum,

$$\sum_{i=1}^{h} \alpha_i(\alpha_i - j) > (n - \alpha_0)(\alpha_0 - j) = 2j(\alpha_0 - j).$$

Then, on account of (9), we have *a fortiori*

$$2j \sum_{i=1}^{h} (\alpha_i - j) > 2j(\alpha_0 - j),$$

and therefore,

$$\sum_{i=1}^{h} (\alpha_i - j) > \alpha_0 - j = n - 3j. \tag{10}$$

Finally, since $\alpha_1 \geqq \alpha_0$,

$$h \geqq 2.* \tag{11}$$

We also have from (10)

$$\sum_{i=1}^{h} \alpha_i > n + (h - 3)j,$$

so that

$$\sum_{i=1}^{h} \alpha_i > n \quad \text{when} \quad h \geqq 3, \tag{12}$$

an inequality which we shall have occasion to use presently.

6. It will be convenient to let the integer $2j$ be a measure of the complexity of the net N. The simplest nets after the linear ones will then be those for which $2j = 1$.† With this convention, it will be shown that the net N can always be simplified (and therefore finally reduced to a net of lines) by a

* This is one way of showing that the sum of the multiplicities of O_0, O_1, and O_2 is greater than $\alpha_0 + 2j = n$.

† These are the nets defining the so-called Jonquières transformations. They have a base point of order $\nu - 1$ and $2\nu - 2$ simple base points. The proof from now on is a direct generalization of Segre's method for reducing Jonquières nets.

series of quadratic transformations. In simplifying the net, we shall make use only of quadratic transformations of the involutoric type with three distinct base points A, B, and C, each of which is carried into the line through the other two. Moreover, we shall always choose the point O_0 as the base point A, so that the pencil of lines through O_0 will at each stage be transformed into itself. Under these conditions, the complexity $2j$ of each successive net will remain invariant just as long as the point O_0 remains one of the points of highest multiplicity, being measured by the number of variable points of intersection of a line through O_0 with a curve of the net. If, however, the point O_0 ever ceases to be the point of highest multiplicity, the net N' appearing at that stage will be simpler than N, for its complexity will be measured by $2j'$, the number of free points of intersection with N' of a line through a point of higher multiplicity than O_0. It is evident that $2j'$ is less than $2j$.

7. Suppose to begin with that two points O_x and O_y of the set

$$(O_1, O_2, \cdots O_h) = [O_i]_1^h$$

of multiplicities greater than j are distinct from one another and from O_0. They can then be dispersed by a quadratic transformation with O_0, O_x, and O_y as base points. Furthermore, the two new base points which are ordinarily introduced at O_x and O_y are certainly each of multiplicities less than j, because the lines $O_0 O_x$ and $O_0 O_y$ meet the curves of N in at most $n - \alpha_0 - \alpha_h < j$ points. A single transformation will therefore reduce by two the number of base points of multiplicities greater than j. Moreover, if two of the remaining points of the set $[O_i]_1^h$ are distinct from one another and from O_0, we can disperse them in turn by a second transformation, and so on, until less than two remain. But the net N' appearing at this stage must now be simpler than N, for if $2j'$ were equal to $2j$, we should be led into a contradiction with (11). In the particular case where $2j$ is unity, the net N' must consist of straight lines.

8. There remains to be considered the case where at some stage, which we can suppose to be the first, no two points of the set $[O_i]^h$ are distinct from one another and from O_0. Our first quadratic transformation will then be made with one base point at O_0, and the two others A and B in general position. Instead of dispersing any of the points $[O_i]_1^h$, this transformation will introduce two new ones of multiplicities $2j > j$, one at A and one at B. On the other hand, it will detach from O_0 all its infinitely neighboring base points, transforming them into points on and infinitely neighboring to the line AB. If after the transformation there is a point O'' of $[O_i]^{h+2}$ infinitely neighboring to another O', we shall apply a second quadratic transformation with base points at O_0, O', and some third point C in general position. This

one will disperse O', putting O'' in evidence somewhere along OC, but at the same time it will introduce at O' a new base point of multiplicity $2j$. It will therefore not alter the number of points in the set $[O_i]_1^{h+2}$ while it will increase by one the number of distinct ones. Since the line $O_0 O'$ is a fundamental line of the transformation, the last statement would not be true if there could be points of $[O_i]^{h+2}$ other than O' along $O_0 O'$, nor would it be possible to put O'' in evidence it it were infinitely neighboring to O' along $O_0 O'$; but neither of these cases can arise, since the sum of the multiplicities of any two points of $[O_i]^{h+2}$ exceeds $2j$. By a finite number of steps like the last, we can therefore pass to a net with only $h' = h + 2$ points other than O_0 of multiplicities greater than j, all of which will be distinct. We then observe by (12), where h' now replaces h in the formula, that the points $[O_i]^{h'}$ are not all collinear, so that the next quadratic transformation can be made to disperse a pair of them while leaving at least two others distinct from one another and from O_0. A second pair can therefore be dispersed at the next step, after which there will be two less points in the set than there were at the very start. Repeating the whole process a sufficient number of times, we can thus cut down the number of points of the set $[O_i]$ to less than two, when we shall be back to the case previously considered. It is therefore clear that any net may be simplified by successive stages and finally reduced to a net of straight lines.

PRINCETON UNIVERSITY,
November, 1915

WEIERSTRASS'S NON-DIFFERENTIABLE FUNCTION

BY

G. H. HARDY

CONTENTS

1. Introduction 301
2. Weierstrass's function when b is an integer 304
3. Weierstrass's function when b is not an integer 314
4. Other functions 320
4.1. A function which does not satisfy a Lipschitz condition of any order 320
4.2. On a theorem of S. Bernstein 321
4.3. Riemann's non-differentiable function 322

1. Introduction

1.1. It was proved by Weierstrass* that the function

$$f(x) = \sum a^n \cos b^n \pi x, \tag{1.11}$$

where b is an odd integer and

$$0 < a < 1, \tag{1.121}$$

$$ab > 1 + \frac{3}{2}\pi, \tag{1.122}$$

has no differential coefficient for any value of x. Weierstrass's result has been generalized very widely by a number of writers,† who have considered

* Weierstrass, *Abhandlungen aus der Functionenlehre*, p. 97 (see also P. du Bois-Reymond, *Versuch einer Classification der willkürlichen Functionen reeller Argumente nach ihren Änderungen in den kleinsten Intervallen*, Journal für Mathematik, vol. 79 (1875), pp. 21–37).

† I may refer in particular to

Darboux, *Mémoire sur les fonctions discontinues*, Annales de l'École Normale, ser. 2, vol. 4 (1875), pp. 57–112 (pp. 107–108), and vol. 8 (1879), pp. 195–202; Faber, *Einfaches Beispiel einer stetigen nirgends differentiirbaren Funktion*, Jahresbericht der Deutschen Mathematiker Vereinigung, vol. 16 (1907), pp. 538–540; Landsberg, *Über Differentiirbarkeit stetiger Funktionen*, ibid., vol. 17 (1908), pp. 46–51; Lerch, *Über die Nichtdifferentiirbarkeit gewisser Functionen*, Journal für Mathematik, vol. 103 (1888), pp. 126–138;

and to

Bromwich, *Infinite Series*, pp. 490–491; Dini, *Grundlagen*, pp. 205 *et seq.*; Hobson, *Functions of a real variable*, pp. 620 *et seq.*

For a further discussion of certain points concerning Weierstrass's function in particular, see: Wiener, *Geometrische und analytische Untersuchung der Weierstrass'schen Function*, Journal für Mathematik, vol. 90 (1881), pp. 221–252.

I must confess that I have not been able to arrive at a proper understanding of all the contents of this paper.

functions of the general forms

$$C(x) = \sum a_n \cos b_n x, \tag{1.131}$$

$$S(x) = \sum a_n \sin b_n x, \tag{1.132}$$

where the a's and b's are positive, the series $\sum a_n$ is convergent, and the b's increase steadily and with more than a certain rapidity.

A study of the writings to which I have referred, and in particular of the parts of them which bear directly upon Weierstrass's function, soon shows that the last word has not yet been said upon the subject. In this paper I develop a new method for the discussion of this and similar questions, a method less elementary but considerably more powerful than those adopted hitherto. It would be easy to apply it in such a manner as to frame very general conditions for the non-differentiability of the series (1.13). I have not thought it worth while, however, to do this. The interest of my analysis lies, I think, in the method itself and in the results which it gives in a few particularly simple and interesting cases; and its greater power is quite sufficiently illustrated by its application to Weierstrass's classical example.

1.2. The known results concerning Weierstrass's cosine series are, so far as I am aware, as follows. Weierstrass gave the condition

$$ab > 1 + \frac{3}{2}\pi, \tag{1.122}$$

and the only direct improvement that I know on this is Bromwich's

$$ab > 1 + \frac{3}{2}\pi(1 - a). \tag{1.21}$$

These conditions forbid the existence of a differential coefficient *finite or infinite*. For the non-existence of a *finite* differential coefficient there are alternative conditions: Dini's

$$ab \geqq 1, \quad ab^2 > 1 + 3\pi^2, \tag{1.221}$$

Lerch's

$$ab \geqq 1, \quad ab^2 > 1 + \pi^2, \tag{1.222}$$

and finally Bromwich's

$$ab \geqq 1, \quad ab^2 > 1 + \frac{3}{4}\pi^2(1 - a). \tag{1.223}$$

All these conditions presuppose that b is an odd integer. But Dini has also shown that if (1.122) is replaced by

$$ab > 1 + \frac{3}{2}\pi\frac{1 - a}{1 - 3a}, \tag{1.231}$$

or (1.221) by

$$ab \geqq 1, \qquad ab^2 > 1 + 15\pi^2 \frac{1-a}{5-21a}, \tag{1.232}$$

then this restriction may be removed. It is naturally presupposed in (1.231) that $a < \frac{1}{3}$ and in (1.232) that

$$a < \frac{5}{21}.$$

These conditions are all obviously artificial. It would be difficult to believe that any of them really correspond to any essential feature of the problem under discussion. They arise merely in consequence of the limitations of the methods employed. There is in fact only one condition which suggests itself naturally and seems obviously relevant, viz:

$$ab \geqq 1.* \tag{1.24}$$

1.3. The chief results which I prove here concerning Weierstrass's function, and the corresponding function defined by a series of sines, may be summarized as follows. In none of the results is b restricted to be an integer.

THEOREM 1.31. *Neither of the functions*

$$C(x) = \sum a^n \cos b^n \pi x, \qquad S(x) = \sum a^n \sin b^n \pi x,$$

where $0 < a < 1$, $b > 1$, *possesses a finite differential coefficient at any point in any case in which*

$$ab \geqq 1.$$

THEOREM 1.32. *Theorem* 1.31 *becomes untrue if the word "finite" is omitted.*

1.33. *If* $ab > 1$ *and so*

$$\xi = \frac{\log(1/a)}{\log b} < 1,$$

then each of the functions satisfies the condition

$$f(x+h) - f(x) = O(|h|^{\xi}),$$

for every value of x*; but neither of them satisfies*

$$f(x+h) - f(x) = o(|h|^{\xi}),$$

for any value of x.

* Hadamard (*La série de Taylor et son prolongement analytique*, p. 31, f. n.), referring to the sine-series, remarks: "Cette hypothèse ($ab > 1 + \frac{3}{2}\pi$) est celle que nécessite le raisonnement de Weierstrass. En réalité, il suffit que $ab > 1$"; but no proof of this assertion has ever been published, and I understand it as merely the expression of an opinion, the more so since the wording of the remark is ambiguous. It is not stated whether b is merely an integer (as in Hadamard's text), or an odd integer (as in Weierstrass's discussion of the cosine-series), nor whether the condition precludes the existence of any differential coefficient or only of a finite one. If the differential coefficient is not restricted to be finite, the assertion is, as we shall see later, untrue. If it is so restricted, then it is enough that $ab \geqq 1$.

In section 2 I prove these theorems on the assumption that b is integral. In section 3 I extend the results to the general case. In section 4 I give a simple example of a function, represented by an absolutely convergent Fourier series, which does not satisfy a "Lipschitz condition" of any order for any value of x; I introduce a short digression concerning a theorem of S. Bernstein; and I discuss the question of the differentiability of the function

$$f(x) = \sum \frac{\sin n^2 \pi x}{n^2}.$$

This function is of interest for historical reasons, as it was supposed not to be differentiable by Riemann and his pupils.* I prove here that $f(x)$, and indeed the function derived from $f(x)$ by replacing n^2 by n^a, where $\alpha < \frac{5}{2}$, has no finite differential coefficient for any *irrational* value of α. This result lies a good deal deeper than anything else in the paper. The proof which I give is, as it stands, simple, but it depends on previous results established by Mr. Littlewood and myself by reasoning of a highly transcendental character.

2. Weierstrass's function when b is an integer

2.1. I shall suppose throughout this section that b is integral, and I shall write $\pi x = \theta$, so that Weierstrass's series is a Fourier series in θ. I shall begin by proving a series of lemmas.

Suppose that $G(r, \theta)$ is a harmonic function, the real part of a power-series

$$\sum a_n z^n = \sum a_n r^n e^{ni\theta}$$

convergent when $r < 1$. Suppose further that $G(r, \theta)$ is continuous for $r \leqq 1$, and that

$$G(1, \theta) = g(\theta).$$

Lemma 2.11. *If*

$$g(\theta) - g(\theta_0) = o(|\theta - \theta_0|^\alpha),$$

where $0 < \alpha < 1$, *when* $\theta \to \theta_0$, *then*

$$\frac{\partial G(r, \theta_0)}{\partial \theta_0} = o\{(1 - r)^{-(1-\alpha)}\}$$

when $r \to 1$.

It is obvious that we may, without loss of generality, suppose $\theta_0 = 0$. We have

$$G(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r \cos(u - \theta) + r^2} g(u)\, du, \tag{2.111}$$

when $r < 1$. Differentiating with respect to θ, and then putting $\theta = \theta_0 = 0$, we obtain

* See du Bois-Reymond's memoir quoted on p. 301 (p. 28).

$$\frac{\partial G}{\partial \theta_0} = \frac{r(1-r^2)}{\pi}\int_{-\pi}^{\pi}\frac{\sin u}{\Delta^2}g(u)\,du, \tag{2.112}$$

where

$$\Delta = 1 - 2r\cos u + r^2. \tag{2.113}$$

Let

$$g(u) - g(0) = \gamma(u). \tag{2.114}$$

Then we can choose δ so that

$$|\gamma(u)| < \epsilon|u|^a \tag{2.115}$$

for $-\delta \leqq u \leqq \delta$. Also

$$\begin{aligned}\int_{-\pi}^{\pi}\frac{\sin u}{\Delta^2}g(u)\,du &= \int_{-\pi}^{\pi}\frac{\sin u}{\Delta^2}\gamma(u)\,du \\ &= \left(\int_{-\pi}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{\pi}\right)\frac{\sin u}{\Delta^2}\gamma(u)\,du \\ &= J_1 + J_2 + J_3,\end{aligned} \tag{2.116}$$

say. Plainly

$$J_1 = O(1), \qquad J_3 = O(1), \tag{2.117}$$

when $r \to 1$. Also

$$\begin{aligned}|J_2| &< 2\epsilon\int_0^{\delta}\frac{u^{a+1}\,du}{\Delta^2} = 2\epsilon\int_0^{\delta}\frac{u^{a+1}\,du}{\{(1-r)^2 + 4r\sin^2\frac{1}{2}u\}^2} \\ &< 32\epsilon\int_0^{\delta}\frac{u^{a+1}\,du}{\{(1-r)^2 + ru^2\}^2} \overset{*}{<} 32\epsilon\int_0^{\infty}\frac{u^{a+1}\,du}{\{(1-r)^2 + ru^2\}^2} \\ &= 32\epsilon\frac{(1-r)^{a-2}}{r^{1+\frac{1}{2}a}}\int_0^{\infty}\frac{w^{a+1}\,dw}{(1+w^2)^2}.\end{aligned} \tag{2.118}$$

From (2.112), (2.116), (2.117), and (2.118) it follows that

$$\left|\frac{\partial G}{\partial \theta_0}\right| < K\epsilon(1-r)^{-(1-a)}, \tag{2.119}$$

where K is a constant, for all values of r near enough to 1; and the lemma is therefore proved.

LEMMA 2.12. *If $g(\theta)$ possesses a finite differential coefficient $g'(\theta_0)$ for $\theta = \theta_0$, then*

$$\frac{\partial G}{\partial \theta_0} \to g'(\theta_0)$$

when $r \to 1$.

This is a known theorem, due to Fatou.† The principle of the proof does

* Since $1 - r > \frac{1}{2}(1-r)$ and $\sin\frac{1}{2}u \geqq \frac{1}{4}u$.

† *Séries trigonométriques et séries de Taylor*, Acta Mathematica, vol. 30 (1906), pp. 335–400.

not differ essentially from that of Lemma 2.11, and I need hardly repeat it.

LEMMA 2.13. *Suppose that $f(y)$ is a real or complex function of the real variable y, possessing a p-th differential coefficient $f^{(p)}(y)$ continuous throughout the interval $0 < y \leqq y_0$. Suppose further that $\lambda \geqq 0$, that*

$$f(y) = o(y^{-\lambda})$$

if $\lambda > 0$ and

$$f(y) = A + o(1)$$

if $\lambda = 0$, and in either case that

$$f^{(p)}(y) = O(y^{-p-\lambda}).$$

Then

$$f^{(q)}(y) = o(y^{-q-\lambda})$$

for $0 < q < p$.

This is a special case of a result proved by Mr. Littlewood and myself in 1912.*

LEMMA 2.14. *If $\rho > 0$ and*

$$f(y) = \sum b^{n\rho} e^{-b^n y}$$

then

$$f(y) = O(y^{-\rho})$$

as $y \to 0$.

For, if we write

$$e^{-y} = u, \qquad f(y) = \sum a_n u^n, \tag{2.141}$$

$$a_0 + a_1 + \cdots + a_n = s_n, \tag{2.142}$$

then

$$\begin{aligned} s_n &= 1 + b^\rho + b^{2\rho} + \cdots + b^{\nu\rho} \\ &= O(b^{\nu\rho}) = O(n^\rho) \end{aligned} \tag{2.143}$$

for

$$b^\nu \leqq n < b^{\nu+1}.$$

From (2.141), (2.142), and (2.143) it follows that

$$f(y) = (1-u)\sum s_n u^n = O\{(1-u)\sum n^\rho u^n\} = O\{(1-u)^{-\rho}\} = O(y^{-\rho}).$$

LEMMA 2.15. *If*

$$\sin b^n \pi x \to 0$$

as $n \to \infty$, then

$$x = \frac{p}{b^q}$$

where p and q are integers; so that $\sin b^n \pi x = 0$ for $n \geqq q$.

* Hardy and Littlewood, *Contributions to the arithmetic theory of series*, Proceedings of the London Mathematical Society, ser. 2, vol. 11 (1912), pp. 411–478. The result required is obtained from Theorems 6 and 8 (*l. c.*, pp. 426–427) by supposing

$$\phi = \psi = y^{-\lambda}.$$

We have

$$b^n x = k_n + \epsilon_n,$$

where k_n is an integer and $\epsilon_n \to 0$. It follows that

$$bk_n - k_{n+1} + b\epsilon_n - \epsilon_{n+1} = 0;$$

and it is evident that this is only possible if

$$k_{n+1} = bk_n, \qquad \epsilon_{n+1} = b\epsilon_n$$

from a certain value of n onwards, say for $n \geqq \nu$. We have therefore

$$\epsilon_{\nu+\mu} = b^\mu \epsilon_\nu$$

and so

$$b^\mu \epsilon_\nu \to 0$$

as $\mu \to \infty$. This is only possible if $\epsilon_\nu = 0$, which proves the lemma.

2.2. The proof of the final lemma is a little more elaborate. In stating it I use a notation introduced by Mr. Littlewood and myself:* I write

$$f = \Omega(\phi)$$

as signifying *the negation of* $f = o(\phi)$, that is to say as asserting the existence of a constant K such that

$$|f| > K\phi$$

for some special sequence of values whose limit is that to which the variable is supposed to tend.

LEMMA 2.21. *Suppose that*

$$f(y) = \sum b^{n\rho} e^{-b^n y} \sin b^n \pi x,$$

where $y > 0$, *and that*

$$x \neq \frac{p}{b^q}$$

for any integral values of p *and* q. *Then*

$$f(y) = \Omega(y^{-\rho})$$

for all sufficiently large values of ρ.

I consider a special sequence of values of y, viz.,

$$y = \frac{\rho}{b^m} \qquad (m = 1, 2, 3, \cdots); \tag{2.211}$$

and I write

$$f(y) = \sum u_n = \sum_0^{m-1} u_n + u_m + \sum_{m+1}^{\infty} u_n = f_1 + f_2 + f_3, \tag{2.212}$$

* Hardy and Littlewood, *Some problems of diophantine approximation* (II), Acta Mathematica, vol. 37 (1914), pp. 193–238 (p. 225).

say. We have in the first place

$$
\begin{aligned}
|f_1| &< b^{m\rho}\, e^{-b^m y}\{b^{-\rho}\, e^{(b^m - b^{m-1})y} + b^{-2\rho}\, e^{(b^m - b^{m-2})y} + \cdots\} \\
&= b^{m\rho}\, e^{-\rho}\{e^{-\rho[\log b - 1 + (1/b)]} + e^{-\rho[2\log b - 1 + (1/b^2)]} + \cdots\} \\
&< b^{m\rho}\, e^{-\rho}\frac{e^{-B\rho}}{1 - e^{-B\rho}},^{*}
\end{aligned}
\tag{2.2131}
$$

where

$$
B = \log b - 1 + \frac{1}{b}. \tag{2.2132}
$$

Similarly

$$
\begin{aligned}
|f_3| &< b^{m\rho}\, e^{-b^m y}\{b^{\rho}\, e^{-(b^{m+1} - b^m)y} + b^{2\rho}\, e^{-(b^{m+2} - b^m)y} + \cdots\} \\
&= b^{m\rho}\, e^{-\rho}\{e^{-\rho(b - 1 - \log b)} + e^{-\rho(b^2 - 1 - 2\log b)} + \cdots\} \\
&< b^{m\rho}\, e^{-\rho}\frac{e^{-B'\rho}}{1 - e^{-B'\rho}},^{\dagger}
\end{aligned}
\tag{2.2141}
$$

where

$$
B' = b - 1 - \log b. \tag{2.2142}
$$

Now it follows from Lemma 2.15 that $\sin b^m \pi x$ does not tend to zero. There is therefore a positive constant c such that

$$
|\sin b^m \pi x| > c \tag{2.215}
$$

for an infinity of values of m, and we shall confine our attention to such values.

We can choose ρ_0 so that

$$
\frac{e^{-B\rho}}{1 - e^{-B\rho}} + \frac{e^{-B'\rho}}{1 - e^{-B'\rho}} < \tfrac{1}{2}c \tag{2.216}
$$

for $\rho > \rho_0$. Using (2.212), (2.2131), (2.2141), (2.216), and (2.215), we obtain:

$$
\begin{aligned}
|f| &\geqq |f_2| - |f_1| - |f_3| \\
&> b^{m\rho}\, e^{-\rho}\,(|\sin b^m \pi x| - \tfrac{1}{2}c) \\
&> Kb^{m\rho} > Ky^{-\rho},
\end{aligned}
$$

where the K's are constants, for $\rho > \rho_0$ and an infinity of values of y tending to zero. This completes the proof of the lemma.

* Here we use the inequalities

$$
B = \log b - 1 + \frac{1}{b} > 0
$$

and

$$
m \log b - 1 + \frac{1}{b^m} > m\left(\log b - 1 + \frac{1}{b}\right).
$$

† Here we use the inequalities

$$
B' = b - 1 - \log b > 0
$$

and

$$
b^m - 1 - m \log b > m\,(b - 1 - \log b).
$$

2.3. We can now proceed to the proof of our main results. Let us suppose first that

$$ab > 1, \tag{2.311}$$

$$x \neq \frac{p}{b^q}, \tag{2.312}$$

and that

$$f(x) = \sum a^n \cos b^n \pi x = \sum a^n \cos b^n \theta = g(\theta) \tag{2.321}$$

satisfies the condition

$$f(x+h) - f(x) = o(|h|^\xi) \tag{2.3221}$$

or, what is the same thing

$$g(\theta + h) - g(\theta) = o(|h|^\xi), \tag{2.3222}$$

where

$$\xi = \frac{\log(1/a)}{\log b} < 1. \tag{2.323}$$

Then, if

$$G(r, \theta) = \sum a^n r^{b^n} \cos b^n \theta = \sum a^n e^{-b^n y} \cos b^n \pi x, \tag{2.33}$$

we have, by Lemma 2.11,

$$\begin{aligned} F(y) = \frac{\partial G}{\partial \theta} &= -\sum (ab)^n e^{-b^n y} \sin b^n \pi x \\ &= -\sum b^{(1-\xi)n} e^{-b^n y} \sin b^n \pi x \\ &= o(y^{\xi-1}) \end{aligned} \tag{2.34}$$

when $r \to 1$, $y \to 0$. We have also, by Lemma 2.14,

$$\begin{aligned} F^{(p)}(y) &= (-1)^{p+1} \sum (ab^{p+1})^n e^{-b^n y} \sin b^n \pi x \\ &= O \sum b^{(p+1-\xi)n} e^{-b^n y} \\ &= O(y^{\xi-p-1}), \end{aligned} \tag{2.35}$$

for all positive values of p. It follows, by Lemma 2.13, that

$$F^{(q)}(y) = o(y^{\xi-q-1}) \tag{2.36}$$

for $0 < q < p$, and therefore for all positive values of q. But this contradicts Lemma 2.21, if q is sufficiently large. The conditions (2.322) can therefore not be satisfied.

The case in which

$$ab = 1, \qquad \xi = 1, \tag{2.37}$$

may be treated in the same manner. The only difference is that we use Lemma 2.12 instead of Lemma 2.11, and that our final conclusion is that $f(x)$

cannot possess a finite differential coefficient for any value of x which is not of the form p/b^q.

2.4. This reasoning fails when $x = p/b^q$, and such values of x require special examination. We have in this case

$$\cos\{b^n\pi(x+h)\} = \cos(b^{n-q}p\pi + b^n\pi h) = \pm\cos b^n\pi h$$

for $n > q$, the negative sign being taken if b and p are both odd and the positive sign otherwise. The properties of the function in the neighborhood of such a value of x are therefore the same, for our present purpose, as those of the function

$$f(h) = \sum a^n \cos b^n\pi h \tag{2.41}$$

near $h = 0$. Now

$$f(h) - f(0) = -2\sum a^n \sin^2 \tfrac{1}{2}b^n\pi h$$

$$= -2(f_1 + f_2),$$

where

$$f_1 = \sum_0^\nu a^n \sin^2 \tfrac{1}{2}b^n\pi h, \qquad f_2 = \sum_{\nu+1}^\infty a^n \sin^2 \tfrac{1}{2}b^n\pi h.$$

Choose ν so that

$$b^\nu|h| \leqq 1 < b^{\nu+1}|h|. \tag{2.42}$$

Then

$$f_1 + f_2 > f_1 > \sum_0^\nu a^n(b^n h)^2 = h^2\frac{(ab^2)^{\nu+1} - 1}{ab^2 - 1}$$

$$> Kh^2(ab^2)^\nu > Ka^\nu > Kb^{-\xi\nu} > K|h|^\xi,$$

where the K's are constants. It follows that

$$f(h) - f(0) \neq o(|h|^\xi).$$

If $ab > 1$, $\xi < 1$, we have proved that we want. In this case the graph of $f(h)$ has a cusp (pointing upwards) for $h = 0$, and that of Weierstrass's function has a cusp for $x = p/b^q$. If on the other hand $ab = 1$, $\xi = 1$, then we have proved that

$$\overline{\lim_{h\to+0}}\frac{f(h) - f(0)}{h} < 0, \qquad \underline{\lim_{h\to-0}}\frac{f(h) - f(0)}{h} > 0,$$

so that $f(h)$ has certainly no finite differential coefficient for $h = 0$, nor Weierstrass's function for $x = p/b^q$.

2.5. We have thus proved Theorems 1.31 and 1.33 in so far as they relate to the cosine-series and are of a negative character.

We have next to prove that, when $\xi < 1$, Weierstrass's function satisfies the condition

$$f(x+h) - f(x) = O(|h|^\xi)$$

for all values of x. We have

$$f(x+h)-f(x) = -2\sum a^n \sin\{b^n\pi(x+\tfrac{1}{2}h)\}\sin\tfrac{1}{2}b^n\pi h$$
$$= O\sum a^n|\sin\tfrac{1}{2}b^n\pi h|.$$

Choose ν as in (2.42). Then

$$f(x+h)-f(x) = O\Big(|h|\sum_0^{\nu} a^n b^n + \sum_{\nu+1}^{\infty} a^n\Big)$$
$$= O(a^\nu b^\nu|h|+a^\nu)$$
$$= O(a^\nu) = O(|h|^\xi).$$

The condition is therefore satisfied, and indeed uniformly in x. It should be observed that our argument fails when $ab = 1$, $\xi = 1$. In this case we can only assert that

$$f(x+h)-f(x) = O(\nu|h|+a^\nu) = O\left(|h|\log\frac{1}{|h|}\right).$$

It should also be observed that the argument of this paragraph applies to the sine series as well as to the cosine series, and is independent of the restriction that b is an integer.

2.6. The proof of Theorems 1.31 and 1.33 is now complete so far as the cosine series is concerned. The corresponding proof for the sine-series differs only in detail. The lemmas required are the same except that Lemma 2.15 must be replaced by

LEMMA 2.61. *If*

$$\cos b^n\pi x \to 0,$$

then b must be odd and

$$x = \frac{p+\frac{1}{2}}{b^q};$$

so that $\cos b^n\pi x = 0$ *from a certain value of n onwards:*

and that corresponding changes must be made in the wording of Lemma 2.21.

If now the value of x is not exceptional (i. e. one of those specified in Lemma 2.61), we can repeat the arguments of 2.3. It is therefore only necessary to discuss the exceptional values, which can exist only if b is odd. We have in this case

$$\sin\{b^n\pi(x+h)\} = \sin(b^{n-q}p\pi + \tfrac{1}{2}b^{n-q}\pi + b^n\pi h)$$
$$= \pm\sin(\tfrac{1}{2}b^{n-q}\pi + b^n\pi h),$$

for $n > q$, the sign being fixed as in 2.4. The last function is numerically equal to $\cos b^n\pi h$; it has always the same sign as $\cos b^n\pi h$, or always the opposite sign, if b is of the form $4k+1$; while if b is of the form $4k+3$ the

signs agree and differ alternately. The problem is therefore reduced, either to the discussion of the function (2.41) near $h = 0$ (a discussion made already), or to that of

$$f(h) = \sum (-a)^n \cos b^n \pi h.$$

We have to prove that

$$f(h) - f(0) \neq o(|h|^\xi)$$

if $\xi < 1$, and that $f(h)$ has not a finite differential coefficient for $h = 0$, if $\xi = 1$.

I consider the special sequence of values

$$h = \frac{2}{b^\nu} \qquad (\nu = 1, 2, 3, \cdots).$$

We have

$$f(h) - f(0) = -2\sum_0^{\nu-1} (-a)^n \sin^2 \tfrac{1}{2} b^n \pi h$$

$$= (-1)^\nu 2a^{\nu-1} \sum_1^\nu \left(-\frac{1}{a}\right)^n \sin^2 \frac{\pi}{b^n}.$$

Now

$$\sum_1^\nu \left(-\frac{1}{a}\right)^n \sin^2 \frac{\pi}{b^n} \to \sum_1^\infty \left(-\frac{1}{a}\right)^n \sin^2 \frac{\pi}{b^n} = S,$$

say; and S being the sum of an alternating series of decreasing terms,* is positive.

Also

$$a^\nu = b^{-\xi\nu} = (\tfrac{1}{2}h)^\xi.$$

Thus $f(h) - f(0)$ is, for the particular sequence of values in question, greater in absolute value than a constant multiple of h^ξ, and alternately positive and negative. This completes the proof of Theorems 1.31 and 1.33.

* In order to prove this we must show that

$$\sin^2 \frac{\pi}{b^n} > \frac{1}{a} \sin^2 \frac{\pi}{b^{n+1}};$$

and this will certainly be true if

$$f(\theta) = \sin^2 \theta - b \sin^2 \frac{\theta}{b}$$

is positive for $0 < \theta \leqq \pi/b$. We have

$$f'(\theta) = \sin 2\theta - \sin \frac{2\theta}{b} = 2 \sin \frac{b-1}{b} \theta \cos \frac{b+1}{b} \theta,$$

which is positive for

$$0 < \theta < \frac{b\pi}{2(b+1)}.$$

Since

$$\frac{b}{2(b+1)} > \tfrac{1}{3}$$

if $b > 2$, this proves what we want, when $b \geqq 3$. When $b = 2$, we observe that $f(\theta)$ decreases from $\theta = \frac{1}{3}\pi$ to $\theta = \frac{1}{2}\pi$, and that

$$f(\tfrac{1}{3}\pi) = 1 - 2\left(\frac{1}{\sqrt{2}}\right)^2 = 0.$$

2.7. The possibility of the existence of a finite differential coefficient is thus disposed of in all cases. The question remains whether an equally comprehensive result holds for *infinite* differential coefficients. The theorem which follows, which includes theorem 1.32, shows that the answer to this question is negative.

THEOREM 2.71. *If*

$$ab \geqq 1, \qquad a(b+1) < 2$$

then the sine-series has the differential coefficient $+\infty$ *for* $x = 0$; *and if* b *is of the form* $4k+1$, *then the same is true of the cosine series for* $x = \frac{1}{2}$.

It is enough to prove the first of these statements, the second then following immediately by the transformation $x = \frac{1}{2} + y$.

We have

$$\frac{f(h)-f(0)}{h} = \frac{1}{h}\sum a^n \sin b^n \pi h \tag{2.711}$$

$$= \frac{1}{h}\sum_0^{\nu-1} a^n \sin b^n \pi h + \frac{1}{h}\sum_\nu^\infty a^n \sin b^n \pi h$$

$$= f_1 + f_2,$$

say, where ν is chosen so that

$$b^{\nu-1}|h| \leqq \tfrac{1}{2} < b^\nu |h|. \tag{2.712}$$

Suppose first that $ab > 1$. Then

$$f_1 > 2\sum_0^{\nu-1}(ab)^n = 2\frac{(ab)^\nu - 1}{ab-1}, \tag{2.7131}$$

$$|f_2| < \frac{1}{|h|}\sum_\nu^\infty a^n = \frac{a^\nu}{(1-a)|h|}. \tag{2.7132}$$

Now

$$a(b+1) < 2, \qquad 1-a > ab-1;$$

so that

$$\frac{1-a}{ab-1} = 1+\delta, \tag{2.714}$$

where $\delta > 0$. We can suppose h so small (or ν so large) that

$$\frac{(ab)^\nu - 1}{(ab)^\nu} > \frac{2+\delta}{2(1+\delta)}. \tag{2.715}$$

Then from (2.7131), (2.7132), (2.712), (2.714), and (2.715) it follows that

$$\frac{|f_2|}{f_1} < \frac{(ab)^\nu}{(ab)^\nu - 1}\,\frac{ab-1}{1-a} < \frac{1}{1+\frac{1}{2}\delta};$$

and so that $f_1 + f_2$ is greater than a constant multiple of f_1 or of $(ab)^\nu$. Thus

$$\frac{f(h)-f(0)}{h} \to +\infty \tag{2.716}$$

as $h \to 0$.

If on the other hand $ab = 1$, then $|f_2|$ remains less than a constant and

$$f_1 > 2\nu \to +\infty,$$

so that (2.716) is still true.

When b is given, a number $\alpha(b)$ exists which is the *least* number such that the condition

$$ab > \alpha(b)$$

forbids the existence of a differential coefficient finite or infinite. All that we can say about $\alpha(b)$ at present is that

$$\frac{2}{b+1} \leqq \alpha(b) \leqq \frac{1+\frac{3}{2}\pi}{b+\frac{3}{2}\pi}.^*$$

2.8. We have now proved everything that was stated in Section 1, subject to the restriction that b is an integer.

When b is not an integer, our series are no longer Fourier's series, and we can no longer employ Poisson's integral (2.111). The first stage in the discussion is naturally to construct a new formula to replace Poisson's. When we have done this, we find that some further modifications of the argument are needed, owing to the lack of any simple result corresponding to Lemmas 2.15 and 2.61, and the difficulty of determining precisely the exceptional values of x for which $\sin b^n \pi x \to 0$ or $\cos b^n \pi x \to 0$. It will be found, however, that no fundamental change in the method is necessary, and that the additional analysis required is not of an elaborate character.

3. Weierstrass's function when b is not an integer

3.1. I suppose now that b is any number greater than 1; and I write

$$s = \sigma + it, \tag{3.11}$$

(as is usual in the theory of Dirichlet's series), and

$$f(s) = \sum_1^\infty a^n e^{-b^n s} = G(\sigma, t) + iH(\sigma, t) \qquad (\sigma \geqq 0), \tag{3.121}$$

$$G(0, t) = g(t). \tag{3.122}$$

3.2. Lemma 3.2. *If $\sigma > 0$ then*

$$G(\sigma, t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sigma g(u)}{\sigma^2 + (u-t)^2}\, du.$$

* The first inequality follows from 2.71, the second from (1.21).

We have, by a well-known formula,*

$$G(\sigma_0, t_0) = \frac{1}{2\pi}\int_C \left(\log\frac{r}{r'}\frac{dG}{dn} - G\frac{d}{dn}\log\frac{r}{r'}\right) dS, \tag{3.21}$$

where

$$G = G(\sigma, t),$$

$$r = \sqrt{(\sigma-\sigma_0)^2 + (t-t_0)^2},$$

$$r' = \sqrt{(\sigma+\sigma_0)^2 + (t-t_0)^2},$$

C is a closed contour which lies entirely in the half-plane $\sigma > 0$ and includes the point (σ_0, t_0) in its interior, and dS and dn are elements of the arc of C and the outward normal to C.

I take C to be the rectangle whose vertices are the points

$$(\delta, T), \quad (\delta, -T), \quad (\beta, -T), \quad (\beta, T)$$

where

$$0 < \delta < \beta, \qquad T > 0,$$

and I denote the four sides of the rectangle, taken in order from the vertex (δ, T), by C_1, C_2, C_3, C_4, and the corresponding parts of the integral by J_1, J_2, J_3, J_4.

Suppose first that $\delta \to 0$. The functions G and

$$-\frac{\partial}{\partial\sigma}\log\frac{r}{r'} = -\frac{\sigma-\sigma_0}{(\sigma-\sigma_0)^2+(t-t_0)^2} + \frac{\sigma+\sigma_0}{(\sigma+\sigma_0)^2+(t-t_0)^2}$$

are continuous for $0 \leqq \sigma \leqq \delta$, $-T \leqq t \leqq T$. We have also, when $\sigma = \delta$,

$$\log\frac{r}{r'} = \log\sqrt{\frac{(\sigma_0-\delta)^2+(t-t_0)^2}{(\sigma_0+\delta)^2+(t-t_0)^2}} = O(\delta),$$

uniformly for $-T \leqq t \leqq T$; and

$$\frac{dG}{dn} = -\frac{\partial G}{\partial\sigma} = \sum (ab)^n e^{-b^n\sigma}\cos b^n t = O\sum (ab)^n e^{-b^n\sigma},$$

which is of the form

$$O(\sigma^{\xi-1}) \qquad (0 < \xi < 1)$$

if $ab > 1$, and of the form

$$O\left(\log\frac{1}{\sigma}\right)$$

if $ab = 1$. From these facts it follows at once that

$$J_1 \to \mathcal{L}_1 = \frac{1}{\pi}\int_{-T}^{T} \frac{\sigma_0\, g(t)}{\sigma_0^2 + (t-t_0)^2}\, dt,$$

* See for example Picard, *Traité d'analyse*, vol. 2, pp. 15, 16.

and that J_2 and J_4 tend to limits $\mathfrak{L}_2$ and $\mathfrak{L}_4$, the former of these limits, for example, being given by the formula

$$\mathfrak{L}_2 = \frac{1}{2\pi}\int_0^\beta \left(\log\frac{r}{r'}\frac{dG}{dn} - G\frac{d}{dn}\log\frac{r}{r'}\right)d\sigma,$$

wherein $t = -T$.

Suppose next that $T \to \infty$. Then we have, in $\mathfrak{L}_3$,

$$G = O(1), \qquad \frac{dG}{dn} = -\frac{\partial G}{\partial t} = \frac{O(1)}{\sigma^{1-\xi}}*,$$

$$\log\frac{r}{r'} = O\left(\frac{1}{T^2}\right), \qquad \frac{d}{dn}\log\frac{r}{r'} = -\frac{\partial}{\partial t}\log\frac{r}{r'} = O\left(\frac{1}{T^3}\right),$$

uniformly for $0 \leqq \sigma \leqq \beta$. Thus $\mathfrak{L}_2 \to 0$, and similarly $\mathfrak{L}_4 \to 0$. Also $\mathfrak{L}_1$ and J_3 obviously tend to limits $\mathfrak{L}$ and $\mathfrak{H}$. The formula (3.21) thus passes over into

$$G(\sigma_0, t_0) = \mathfrak{L} + \mathfrak{H}, \tag{3.22}$$

where

$$\mathfrak{L} = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sigma_0\, g(t)}{\sigma_0^2 + (t-t_0)^2}dt, \tag{3.23}$$

$$\mathfrak{H} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\log\frac{r}{r'}\frac{dG}{dn} - G\frac{d}{dn}\log\frac{r}{r'}\right)dt, \tag{3.24}$$

the value of σ in $\mathfrak{H}$ being β.

Finally we make $\beta \to \infty$. If we observe that

$$\log\frac{r}{r'} = O\left(\frac{1}{T^2}\right), \qquad \frac{d}{dn}\log\frac{r}{r'} = \frac{\partial}{\partial\sigma}\log\frac{r}{r'} = O\left(\frac{1}{T^2}\right),$$

uniformly for $\sigma \geqq \beta$; and that

$$G = O(e^{-b\sigma}), \qquad \frac{dG}{dn} = \frac{\partial G}{\partial\sigma} = O(e^{-b\sigma}),$$

uniformly in T; we see that $\mathfrak{H} \to 0$. Thus the proof of the lemma is completed.

3.3. Let us suppose first that $ab > 1$. In this case we shall use, instead of Lemma 2.11, the two lemmas which follow.

LEMMA 3.31. *If*

$$g(t) - g(t_0) = o(|t - t_0|^\alpha),$$

where $0 < \alpha < 1$, *when* $t \to t_0$, *then*

* This must be replaced by

$$O(1)\log\frac{1}{\sigma}$$

when $ab = 1$.

$$\frac{\partial G(\sigma, t_0)}{\partial t_0} = o(\sigma^{a-1}),$$

when $\sigma \to 0$.

LEMMA 3.32. *Under the same conditions*

$$\frac{\partial G(\sigma, t_0)}{\partial \sigma} = o(\sigma^{a-1}).$$

The proofs of these lemmas are very similar, and the first is in all essentials the same as that of Lemma 2.11. It will therefore be sufficient to give the proof of the second.

We may take $t_0 = 0$.* We have then

$$\frac{\partial G}{\partial \sigma} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{t^2 - \sigma^2}{(t^2+\sigma^2)^2} g(t)\, dt. \tag{3.321}$$

If we write

$$g(t) - g(0) = \gamma(t),$$

and observe that

$$\int_{-\infty}^{\infty} \frac{t^2 - \sigma^2}{(t^2+\sigma^2)^2} dt = 0,$$

we see that

$$\frac{\partial G}{\partial \sigma} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{t^2 - \sigma^2}{(t^2+\sigma^2)^2} \gamma(t)\, dt. \tag{3.322}$$

Choose δ so that

$$|\gamma(t)| < \epsilon |t|^a$$

for $-\delta \leqq t \leqq \delta$. Then, if we write

$$\frac{1}{\pi}\int_{-\infty}^{\infty} = \frac{1}{\pi}\left(\int_{-\infty}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{\infty}\right) = \frac{1}{\pi}(J_1 + J_2 + J_3), \tag{3.323}$$

we have

$$J_1 = O(1), \qquad J_3 = O(1), \tag{3.324}$$

and

$$|J_2| < \epsilon \int_{-\delta}^{\delta} \frac{|t^2 - \sigma^2|}{(t^2+\sigma^2)^2} |t|^a\, dt \tag{3.325}$$

$$< \epsilon \int_{-\infty}^{\infty} \frac{|t^2 - \sigma^2|}{(t^2+\sigma^2)^2} |t|^a\, dt < K\epsilon\sigma^{a-1},$$

where K is a constant. The truth of the lemma follows immediately from (3.322)–(3.325).

*The point $t = 0$ has of course a special character for the particular function $g(t)$ which we are considering. This is in no way relevant to the proof of the lemma, which is, like Lemma 2.11, a proposition in the general theory of functions, the truth of which depends only on the validity of the fundamental integral formula, the facts that $g(t)$ is continuous and bounded, and the special hypothesis of the lemma itself. There is therefore no loss of generality in supposing that $t_0 = 0$.

3.4. Suppose now that

$$g(t+h) - g(t) = o(|h|^{\xi}).$$

Then, by Lemmas 3.31 and 3.32, we have

$$\frac{\partial G}{\partial t} = -\sum (ab)^n e^{-b^n\sigma} \sin b^n t = o(\sigma^{\xi-1})$$

and

$$\frac{\partial G}{\partial \sigma} = -\sum (ab)^n e^{-b^n\sigma} \cos b^n t = o(\sigma^{\xi-1}),$$

and so

$$f(y) = \sum (ab)^n e^{-b^n(\sigma+it)} = o(\sigma^{\xi-1}).$$

We can now obtain a contradiction by following the argument of 2.3. It is only necessary to observe that Lemma 2.13 holds for complex as well as for real functions of a real variable, and to use, instead of Lemma 2.21, the proposition: *if*

$$f(y) = \sum b^{n\rho} e^{-b^n(\sigma+it)} \qquad (\sigma > 0)$$

then

$$f(y) = \Omega(\sigma^{-\rho})$$

for all sufficiently large values of ρ. Since $|e^{-b^nit}| = 1$, there is no longer any question of exceptional values of t.

3.5. Now suppose that $ab = 1$. Instead of Lemma 3.31 we use the following lemma, which corresponds to Lemma 2.12.

LEMMA 3.51. *If* $g(t)$ *possesses a finite differential coefficient* $g'(t_0)$ *for* $t = t_0$, *then*

$$\frac{\partial G(\sigma, t_0)}{\partial t_0} \to g'(t_0)$$

when $\sigma \to 0$.

The proof of this presents no fresh difficulty. But it is not necessarily true that

$$\frac{\partial G(\sigma, t_0)}{\partial \sigma}$$

tends to a limit.* It is therefore necessary to follow a line of argument which differs slightly from that of 3.4.

* The difference arises from the fact that the integral

$$\int^{\infty} \frac{2\sigma t}{(\sigma^2 + t^2)^2} t dt$$

is convergent, but the integral

$$\int^{\infty} \frac{\sigma^2 - t^2}{(\sigma^2 + t^2)^2} t dt$$

divergent. All that we can prove is that

$$\frac{\partial G}{\partial \sigma} = o\left(\log \frac{1}{\sigma}\right),$$

and this is not sufficient for our argument.

LEMMA 3.52. *Under the same conditions as those of Lemma 3.51, we have*

$$\frac{\partial^2 G(\sigma, t_0)}{\partial t_0^2} = o\left(\frac{1}{\sigma}\right).$$

Taking $t_0 = 0$, we find

$$\frac{\partial^2 G}{\partial t_0^2} = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\sigma(3t^2 - \sigma^2)}{(\sigma^2 + t^2)^3} g(t)\, dt.$$

In this equation we can replace $g(t)$ by $\gamma(t)$, since

$$\int_{-\infty}^{\infty} \frac{\sigma(3t^2 - \sigma^2)}{(\sigma^2 + t^2)^3}\, dt = 0;$$

and the proof of the lemma may then be completed by arguments similar to those used in the proof of Lemma 3.32.*

3.6. Suppose now that $g(t)$ possesses a finite differential coefficient $g'(t)$, and write

$$f(\sigma) = \frac{\partial G}{\partial t} = -\sum e^{-b^n\sigma} \sin b^n t.$$

Then, by Lemma 3.51, we have

$$f(\sigma) = g'(t) + o(1)$$

when $\sigma \to 0$. But we have also, by Lemma 2.14:

$$f''(\sigma) = -\sum b^{2n} e^{-b^n\sigma} \sin b^n t = O\left(\frac{1}{\sigma^2}\right),$$

and therefore, by Lemma 2.13,

$$f'(\sigma) = \sum b^n e^{-b^n\sigma} \sin b^n t = o\left(\frac{1}{\sigma}\right). \tag{3.61}$$

On the other hand, by Lemma 3.52, we have

$$\frac{\partial^2 G}{\partial t^2} = -\sum b^n e^{-b^n\sigma} \cos b^n t = o\left(\frac{1}{\sigma}\right). \tag{3.62}$$

From (3.61) and (3.62) it follows that

$$F(\sigma) = \sum b^n e^{-b^n(\sigma + it)} = o\left(\frac{1}{\sigma}\right). \tag{3.63}$$

Also, by Lemma 2.14, we have

$$F^{(p)}(\sigma) = (-1)^p \sum b^{(p+1)n} e^{-b^n(\sigma+it)} = O\left(\frac{1}{\sigma^{p+1}}\right) \tag{3.64}$$

* The critical part of the integral is less than a constant multiple of

$$\epsilon\int_{-\delta}^{\delta} \frac{\sigma\,|\,\sigma^2 - 3t^2\,|}{(\sigma^2 + t^2)^3}\,|\,t\,|\,dt,$$

or than a constant multiple of ϵ/σ.

for all values of p. Hence, as in 2.3, it follows that the O may be replaced by o; and this, as before, leads to a contradiction.

3.7. It is only necessary to add the following remarks. Our argument, throughout this section, has been stated in terms of Weierstrass's cosine series. The same arguments apply to the sine series, as there are now no "exceptional values," and it was only the existence of such values which differentiated the two cases in Section 2. The positive statement in Theorem 1.33 has already been proved, the proof given in 2.5 applying to all values of b. No fresh proof is required of Theorem 1.32. The proofs of the theorems stated in 1.3 are therefore now complete in all cases.

4. Other functions

4.1. **A function which does not satisfy a Lipschitz condition of any order.** It was suggested to me recently by Dr. Marcel Riesz that it would be of interest to have an example of an absolutely convergent Fourier's series whose sum does not satisfy any condition of the type

$$f(x+h) - f(x) = O(|h|^\alpha) \qquad (\alpha > 0)$$

for any value of x. The function

$$f(x) = \sum \frac{\cos b^n \pi x}{n^2}$$

is such a function. It is in fact easy to prove, by the methods used in this paper, that

$$f(x+h) - f(x) \neq o\left(\frac{1}{|\log|h||}\right)^2.$$

I should observe, however, that a somewhat less simple example may be found by merely combining remarks made by Faber and Landsberg in their papers quoted on p. 301. Faber writes

$$F(x) = \sum 10^{-n} \phi(2^{n!}x), \tag{4.11}$$

where $\phi(x)$ is the function of period 1 which is equal to x for $0 \leqq x \leqq \frac{1}{2}$ and to $1 - x$ for $\frac{1}{2} \leqq x \leqq 1$, and he shows that

$$F(x+h) - F(x) \neq O\left(\frac{1}{|\log|h||}\right).$$

Landsberg, on the other hand, uses the expansion of a function, substantially equivalent to $\phi(x)$, in a Fourier's series. We have in fact

$$\phi(x) = \frac{1}{4} - \frac{2}{\pi^2}\sum \frac{\cos 2\nu\pi x}{\nu^2} \qquad (\nu = 1, 3, 5 \cdots).$$

If we substitute this expansion in (4.11), we obtain an expansion of $F(x)$ as an absolutely convergent Fourier series, and so an example of the kind required.

4.2. **On a theorem of S. Bernstein.** In this connection it is natural to allude to an important theorem of S. Bernstein, which may be proved very simply by the use of the ideas of this paper. Bernstein's* theorem is as follows:

If $f(x)$ satisfies a Lipschitz condition of order α, where $\alpha > \frac{1}{2}$, throughout the interval $(0, 1)$, i. e. if

$$|f(x+h) - f(x)| < K|h|^{\alpha},$$

where K is an absolute constant, then the Fourier series of $f(x)$ is absolutely convergent. Also $\frac{1}{2}$ is the least number which possesses this property.

Suppose that $2\pi x = \theta$ and

$$f(x) = g(\theta) = \tfrac{1}{2}a_0 + \sum (a_n \cos n\theta + b_n \sin n\theta);$$

and let

$$G(r, \theta) = \tfrac{1}{2}a_0 + \sum r^n (a_n \cos n\theta + b_n \sin n\theta),$$

if $r < 1$, and $G(1, \theta) = g(\theta)$. Then $G(r, \theta)$ is continuous for $0 \leqq r \leqq 1$, $0 \leqq \theta \leqq 2\pi$.

It follows from a simple modification of Lemma 2.11† that

$$\frac{\partial G}{\partial \theta} = -\sum n r^{n-1} (a_n \sin n\theta - b_n \cos n\theta) = O\{(1-r)^{\alpha-1}\},$$

uniformly in θ. Squaring, and integrating from $\theta = 0$ to $\theta = 2\pi$, we obtain

$$\sum n^2 r^{2n} (|a_n|^2 + |b_n|^2) = O(1-r)^{2\alpha-2}.$$

Hence, by putting $r = 1 - (1/\nu)$, we obtain

$$\sum_1^{\nu} n^2 (|a_n|^2 + |b_n|^2) = O(\nu^{2-2\alpha}),$$

and so, by Schwarz's inequality,

$$\sum_1^{\nu} n (|a_n| + |b_n|) = O(\nu^{3-\alpha}). \tag{4.21}$$

From (4.21) it is easy to deduce that the series

$$\sum_1^{\nu} n^{\beta} (|a_n| + |b_n|)$$

is convergent if $\beta < \alpha - \frac{1}{2}$.

* *Sur la convergence absolue des séries trigonométriques*, Comptes Rendus, June 8, 1914.

† With O in the place of o, and "uniformly" inserted in premises and conclusion.

This establishes the truth of the first part of Bernstein's Theorem (indeed rather more). The truth of the second part may be shown by the example of the function

$$g(\theta) = \sum n^{-b} \cos(n^a + n\theta),$$

where $0 < a < 1$, $0 < b < 1$. In this case $G(r, \theta)$ is the real part of

$$F(z) = F(re^{i\theta}) = \sum n^{-b} e^{in^a} z^n.$$

I have shown elsewhere* that this function is continuous for

$$|z| \leqq 1$$

if

$$\tfrac{1}{2}a + b > 1:$$

and it is not difficult to go further and to show that $g(\theta)$ satisfies a Lipschitz condition of order $\frac{1}{2}a + b - 1$.† Now let α be any number less than $\frac{1}{2}$. Then we can choose numbers a and b, each less than 1, and such that

$$\tfrac{1}{2}a + b - 1 > \alpha.$$

The function $g(\theta)$ then satisfies a Lipschitz condition of order greater than α, but its Fourier series is *not* absolutely convergent.

4.3. **Riemann's non-differentiable function.** It was supposed by Riemann‡

* Hardy, *A theorem concerning Taylor's series*, Quarterly Journal of Mathematics, vol. 44 (1913), pp. 147–160. I take this opportunity of correcting a misprint: on p. 153, lines 1 and 6, *for* $(-z)^{(1-b)/a}$ *read* $(-z)^{-(1-b)/a}$.

† The example shows the importance of the distinction between a Lipschitz condition satisfied *at every point of an interval* and one satisfied (uniformly) *throughout the interval*. Since $f(z)$ is regular save for $z = 1$, $g(\theta)$ satisfies a condition of order 1 for every value of θ that is not a multiple of 2π. And it is easy to prove that, at the point $\theta = 0$, it satisfies a condition of order

$$\beta = \operatorname{Min}\left(1, \frac{\frac{1}{2}a + b - 1}{1 - a}\right).$$

Suppose, *e. g.*, that

$$a = \tfrac{3}{4}, \quad b = \tfrac{7}{8}, \quad \tfrac{1}{2}a + b - 1 = \tfrac{1}{4}, \quad \frac{\frac{1}{2}a + b - 1}{1 - a} = 1.$$

Then the function satisfies a condition of order 1 at every point. But it does not satisfy *throughout the interval* $(0, 2\pi)$ any condition of order greater than $\frac{1}{4}$.

This remark was suggested to me by an observation of Dr. Marcel Riesz, viz. that the functions derived from the expansion of

$$(1 - z)^{\lambda} e^{-1/(1-z)}$$

possess similar peculiarities.

‡ My authority for this statement is du Bois-Reymond, who, in his memoir quoted in Section 1, remarks (*l. c.*, p. 28): " Ist seit einigen Jahren wohl hauptsächlich in Deutschlands mathematischen Kreisen von der Möglichkeit von Functionen ohne Differential-quotienten die Rede, besonders seitdem *Riemannsche* Schüler verkündeten, ihr Lehrer habe von der Reihe mit dem gliede $(\sin p^2 x)/p^2$ die Nichtdifferentiirbarkeit behauptet. Diese Reihe solle für gewisse, in jedem noch so kleinen Intervalle unbegrenzt oft wiederkehrende Werthe von x keinen endlichen bestimmten Differentialquotienten zulassen. Einen Beweis hierfür hat unseres Wissens keiner der *Riemannschen* Schüler zu Papiere gebracht, indessen ist nach

that the function

$$f(x) = \sum \frac{\sin n^2 \pi x}{n^2}$$

has no finite differential coefficient for any one of an everywhere dense set of values of x. No proof or disproof of this assertion has, so far as I know, been published. The question is a much more difficult one than any of those connected with Weierstrass's function, owing to the comparatively slow increase of the sequence n^2. But a combination of the methods used earlier in this paper with certain results proved elsewhere* by Mr. Littlewood and myself has led me to a proof of Riemann's assertion and a good deal more.

Suppose that Riemann's function is differentiable† for a certain value of x. Then, by Lemma 2.12, we have

$$\sum r^{n^2} \cos n^2 \pi x = A + o(1),$$

where A is a constant, as $r \to 1$.

But

$$\sum r^{n^2} \cos n^2 \pi x = \Omega\{(1-r)^{-\frac{1}{4}}\}$$

if x is irrational,‡ and

$$\sum r^{n^2} \cos n^2 \pi x = \Omega\{(1-r)^{-\frac{1}{4}}\}$$

if x is a rational of the form $(2\lambda+1)/2\mu$ or $2\lambda/(4\mu+1)$.§ Thus Riemann's function is certainly not differentiable for any irrational (and some rational) values of x. It is easy, by using Lemma 2.11, instead of Lemma 2.12, to show that Riemann's function cannot satisfy the condition

$$f(x+h) - f(x) = o(|h|^{\frac{1}{2}})$$

for any irrational x.

We can prove more, viz.,

THEOREM 4.31. *Neither of the functions*

$$\sum \frac{\cos n^2 \pi x}{n^\alpha}, \qquad \sum \frac{\sin n^2 \pi x}{n^\alpha},$$

where $\alpha < \frac{5}{2}$, *is differentiable for any irrational value of* x.

einer Mitteilung des Herrn *Weierstrass* die *Riemannsche* Behauptung richtig." I am not clear as to the meaning of the last remark. It may mean that Weierstrass, in some communication now lost, had investigated Riemann's function itself. Or it may mean merely that Weierstrass's example showed that Riemann was right in his general view as to the existence of continuous non-differentiable functions.

* Hardy and Littlewood, *Some problems of diophantine approximation* (II), Acta Mathematica, vol. 37 (1914), pp. 193–238.

† In what follows I use "differentiable" as implying the existence of a finite differential coefficient.

‡ *L. c.*, p. 233.

§ *Ibid.*, p. 195. This last result is trivial: that concerning irrational values lies much deeper.

Suppose, e. g., that the sine-series is differentiable. Then, by Lemma 2.12, we have

$$\sum n^{2-\alpha} r^{n^2} \cos n^2 \pi x = A + o(1),$$

or

$$f(y) = \sum n^{2-\alpha} e^{-n^2 y} \cos n^2 \pi x = A + o(1). \tag{4.311}$$

But

$$f^{(p)}(y) = (-1)^p \sum n^{2p+2-\alpha} e^{-n^2 y} \cos n^2 \pi x$$

$$= O \sum n^{2p+2-\alpha} e^{-n^2 y} = O(y^{-p-\frac{3}{2}+\frac{1}{2}\alpha}).$$

Hence, by the theorem of Mr. Littlewood and myself quoted on p. 323,* we have

$$f^{(q)}(y) = o\{y^{-\frac{q}{p}(p+\frac{3}{2}-\frac{1}{2}\alpha)}\},$$

for $0 < q < p$, and in particular

$$f'(y) = o(y^{-1-\frac{3}{2p}+\frac{\alpha}{2p}}). \tag{4.312}$$

But it is easy to prove that

$$f'(y) = -\sum n^{4-\alpha} e^{-n^2 y} \cos n^2 \pi x = \Omega(y^{-\frac{9}{4}+\frac{\alpha}{2}}).\dagger \tag{4.313}$$

From (4.312) and (4.313) it follows that

$$1 + \frac{3}{2p} - \frac{\alpha}{2p} > \frac{9}{4} - \frac{\alpha}{2},$$

and this is impossible if $\alpha < (\frac{5}{2})$ and p is sufficiently large.

It follows from Theorem 4.31 that *the series*

$$\sum \frac{\cos n^2 \pi x}{n^\beta}, \qquad \sum \frac{\sin n^2 \pi x}{n^\beta},$$

where $0 < \beta < \frac{1}{2}$, *are not Fourier's series.* For if the first (e. g.) were a Fourier's series, then the sum of the integrated series

$$\sum \frac{\sin n^2 \pi x}{n^{2+\beta}}$$

would be a function of limited total fluctuation, and would therefore be

* Of which Lemma 2.13 is a special case.

† The contrary hypothesis would involve

$$\Sigma e^{-n^2 y} \cos n^2 \pi x = o(y^{-\lambda}),$$

where

$$\lambda = \frac{9}{4} - \frac{\alpha}{2} - \frac{4-\alpha}{2} = \frac{1}{4};$$

and this is false.

differentiable almost everywhere. This result was proved by Mr. Littlewood and myself, in a different manner, in our paper referred to above.*

I may observe in conclusion that it is easy to prove directly that the function

$$f(x) = \sum \frac{\sin n^2 \pi x}{n^\alpha},$$

where $2 < \alpha < (\frac{5}{2})$ has the differential coefficient $+\infty$ for $x = 0$. A similar direct method could no doubt be applied to an everywhere dense set of rational values of x.

* *L. c.*, p. 237.

FINITE GROUPS REPRESENTED BY SPECIAL MATRICES*

BY

G. A. MILLER

The direct object of this paper is to prove that every possible finite group which contains an abelian subgroup of half its own order can be represented by square matrices all of whose elements are equal to zero with the exception of those which appear in one of the diagonals, and these are all ordinary complex numbers. The fact that every finite group which can be represented by such matrices must contain an abelian subgroup of half its own order is at once evident. The proof of the stated theorem may be based upon several interesting theorems relating to a possible choice of the independent generators of an abelian group. As these auxiliary theorems seem to be quite fundamental, we proceed to develop them in a somewhat more general form than would be necessary for the particular application in view.

1. Independent generators of an abelian group corresponding to a set of independent generators of a quotient group

Let G be any abelian group of order p^m, p being a prime number, and let H represent an arbitrary subgroup of G. If $s_1, s_2, \cdots, s_\lambda$ represent a set of independent generators of G which has been so chosen that as many as possible of these generators are found in H, it is evident that the number of the latter generators cannot exceed λ diminished by the number of independent generators of the quotient group G/H, since at least one independent generator of G in every possible set of such generators must correspond to some power, whose exponent is prime to p, of each operator in every possible set of independent generators of this quotient group. It is, however, not always possible to find a set of independent generators of G such that the number of those contained in H is equal to this upper limit. In fact, if G is of order p^4 and has two independent generators s_1, s_2 of orders p and p^3 respectively, and if we let H represent the subgroup of G which is generated by $s_1 s_2^p$, it is evident that G/H is the cyclic group of order p^2 but that it is impossible to find a set of two independent generators of G such that one of them is contained in H.

The preceding considerations may serve to illustrate the following general

* Presented to the Society, January 1, 1916.

theorem: *If G is an abelian group of order p^m and if H is a subgroup of order p^α, then a set of independent generators of G can always be so selected that at most $m - \alpha$ of them are not contained in H whenever G contains more than $m - \alpha$ independent generators. Moreover, it is possible to construct a group having k independent generators and an arbitrary quotient group of order p^α such that the subgroup corresponding to the identity of this quotient group cannot involve more than $k - \alpha$ of the operators in any possible set of independent generators of the group.*

To prove the first part of this theorem it is only necessary to observe that if $t_1, t_2, \cdots, t_\gamma$ represent a set of independent generators of the quotient group G/H, it is possible to select a set of independent generators of G in such a way that all except one of them are contained in the subgroup corresponding to the group generated by $t_1^p, t_2, \cdots, t_\gamma$.* Let s_1 represent this excepted independent generator of G and consider the subgroup of G generated by the remaining independent generators $s_2, \cdots, s_\lambda$ in any possible set involving s_1. If the order of t_1 exceeds p the independent generators of this subgroup can be selected in such a way that either all of them or all of them with the exception of one are contained in the subgroup of G which corresponds to the subgroup generated by $t_1^{p^2}, t_2, \cdots, t_\gamma$ in the given quotient group. As this process can be continued until we reach the identity in the quotient group, the first part of the theorem in question has been established.

To prove the second part of this theorem we shall first let G be the group generated by β independent operators $s_1, s_2, \cdots, s_\beta$ of orders $p, p^3, \cdots, p^{2\beta-1}$ respectively; and assume that the subgroup H is generated by the $\beta - 1$ independent operators $s_1 s_2^p, s_2 s_3^p, \cdots, s_{\beta-1} s_\beta^p$. The quotient group G/H is cyclic and of order p^β. Since every operator of order p contained in H is generated by operators of G whose order exceeds the order of all the operators of H which generate this operator of order p it results that H cannot contain an independent generator of any possible set of independent generators of G.† This proves the second part of the theorem under consideration whenever G/H is cyclic.

The proof of the theorem when G/H is non-cyclic results almost immediately from what precedes. In fact, if this quotient group has a set of independent generators $t_1, t_2, \cdots, t_\gamma$ of orders $p^{\beta_1}, p^{\beta_2}, \cdots, p^{\beta_\gamma}$ respectively, it is possible to construct a G having $\beta_1 + \beta_2 + \cdots + \beta_\gamma$ independent generators such that none of the operators of any one of the possible sets of independent generators of G appears in H. In fact, to construct such a group it is only necessary to employ successively the process explained in the preceding paragraph for each of the independent generators $t_1, t_2, \cdots, t_\gamma$.

* G. A. Miller, Tôhoku Mathematical Journal, vol. 5 (1914), p. 10.

† G. A. Miller, American Journal of Mathematics, vol. 27 (1905), p. 19.

The theorem which has just been established is a generalization of a theorem established by the present writer in the first article cited above. In the same article was established a necessary and sufficient condition that the subgroup H should contain a minimum number of the independent generators of G. In many considerations it is desirable to assume that a set of independent generators has been selected in such a manner that a given subgroup H involves a maximum number of the operators of this set. This maximum number can clearly not exceed the total number of independent generators of the group diminished by the number of independent generators in its quotient group with respect to H. In fact, it results from the preceding considerations that *a necessary and sufficient condition that a subgroup H of an abelian group G of order p^m can contain all the independent generators of G with the exception of one for each invariant of G/H is that the ratios of the orders of all the powers, besides the identity, of the independent generators of this quotient group and the lowest orders of the corresponding operators of G are equal to each other for the different powers of each independent generator of G/H.*

The maximum number of independent generators of an abelian group G of order p^m which can appear in a given subgroup H may be determined as follows: Select a set of independent generators $t_1, t_2, \cdots, t_\gamma$ of the quotient group G/H and select any operator s_α of lowest order which corresponds to the independent generator t_α, $1 \leqq \alpha \leqq \gamma$. The operator s_α clearly belongs to one of the possible sets of independent generators of G. If the order of t_α is p^{α_1} and if $s_\alpha^{p^\beta}$ is one of the operators of lowest order which correspond to $t_\alpha^{p^\beta}$ $(\beta = 1, 2, \cdots, \alpha_1 - 1)$ then a set of independent generators of G can be so selected that s_α is the only one which corresponds to the various powers of t_α which are not equal to the identity.

In general, the least number of independent generators of such a set which correspond to the various powers of t_α which differ from the identity is equal to one plus the number of times that the lowest order of an operator which corresponds to $t_\alpha^{p^\beta}$ is less than the pth power of the lowest order of an operator which corresponds to $t_\alpha^{p^{\beta-1}}$, $\beta = 1, 2, \cdots, \alpha_1 - 1$. By adding these least numbers for the different independent generators $t_1, t_2, \cdots, t_\gamma$ and subtracting the sum thus obtained from the total number of the independent generators of G there results a remainder which is exactly equal to the maximum number of operators belonging to a set of independent generators of G which can occur in the subgroup H.

In the following section it will be necessary to consider automorphisms of order 2 of an abelian group. It is well known that the multiplying group in such an automorphism is simply isomorphic with the quotient group with respect to the subgroup formed by the invariant operators under this automorphism and that each operator of this multiplying subgroup is trans-

formed into its inverse under this automorphism. In particular, *if G is an abelian group of order p^m, p being an odd prime, and if t transforms G according to an automorphism of order 2, then G is the direct product of the subgroup formed by its invariant operators under t and the corresponding quotient group.* As all the operators of this quotient group are transformed into their inverses by t there results the known theorem that in any automorphism of order 2 of an abelian group of odd order it is always possible to select a set of independent generators in such a way that each of these generators is transformed under this automorphism either into itself or into its inverse.*

When the order of the abelian group G is 2^m, and t transforms G according to an automorphism of order 2, the invariant operators under t constitute a subgroup H and each operator of the quotient group G/H is again transformed into its inverse. In this case G is, however, not the direct product of this quotient group and H, since this quotient group must have at least one operator besides the identity in common with H. Hence we cannot always select the independent generators of G in such a way that each one is transformed into a power of itself by t. It results, however, from the preceding considerations that *if t transforms an abelian group G of order 2^m according to an automorphism of order 2 and if H is the subgroup of G formed by its operators which are invariant under t, then a set of independent generators of G can be so chosen that H contains all of them with the exception of at most three for each invariant of G/H.*

When three independent generators of G must correspond to a particular independent generator of G/H and its powers which differ from the identity in an automorphism of order 2 then these three independent generators must have three different orders and these generators can always be so chosen that the one of highest order is transformed under this automorphism into itself multiplied by the one of next lower order. The latter operator is then transformed into its inverse under this automorphism while the operator of lowest order in the given set of three independent generators is transformed into itself multiplied by a power of the independent generator of next to the highest order whose order is equal to that of this operator of lowest order.

In the other extreme case where a set of independent generators of G can be so selected that only one of these generators corresponds to an independent generator in G/H, this independent generator s must be transformed under this automorphism of order 2 into itself multiplied by an operator whose order is either equal to that of s or to just half of this order. The latter case can occur only when the order of G is 2^m. When the order of G is of this form the automorphism of order 2 can be so chosen that this multiplying operator is any operator whose order has either one of the two given values and

* G. A. Miller, These Transactions, vol. 10 (1909), p. 472.

which occurs in the co-set, with respect to H, involving s^{-2}. Finally, when two independent generators of G must correspond to the powers, which differ from the identity, of an independent generator in G/H, the order of the larger of these two generators is at least four times that of the smaller. If the latter cannot be so chosen that it is the multiplier of the former under this automorphism then the order of this multiplier is either exactly half the order of the former independent generator or twice the order of the latter.

2. Special matrices

To prove the theorem in question by means of the developments of the preceding section it seems desirable to direct attention to the following elementary facts. A square matrix in which each of the elements, with a possible exception of those of the principal diagonal, is equal to zero is called a *quasi-scalar matrix*. Since the product of two such matrices of the same order is a similar matrix in which each element of the principal diagonal is the product of the corresponding elements of the two factors, it results directly that a quasi-scalar matrix whose elements are ordinary complex numbers cannot represent an operator of a finite group unless each of its elements is either zero or a root of unity. In what follows it will be assumed that each of the elements of the principal diagonals of the quasi-scalar matrices under consideration is a root of unity.

A finite set of such quasi-scalar matrices of the same order generates an abelian group of finite order whose rank cannot exceed the common rank of these matrices. Moreover, the operators of every abelian group of rank r can be represented by a set of quasi-scalar matrices of order r. In particular, each operator of a given set of independent generators of an arbitrary abelian group of rank r can be represented by a matrix of order r having all the elements of its principal diagonal, with the exception of one, equal to unity while this one is a root of unity with an index equal to the order of the independent generator which it is to represent, and different independent generators are represented by differently located elements in such matrices.

A square matrix whose secondary diagonal is composed of ordinary complex numbers, none of which is zero, while each of the other elements is zero, will be called a *secondary diagonal matrix*. The product of two such matrices of order n

$$\begin{bmatrix} 0 & 0 & \cdots & a_1 \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & a_{n-1} & \cdots & 0 \\ a_n & 0 & \cdots & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & \cdots & b_1 \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & b_{n-1} & \cdots & 0 \\ b_n & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} a_1 b_n & 0 & \cdots & 0 \\ 0 & a_2 b_{n-1} & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & a_n b_1 \end{bmatrix}$$

is a quasi-scalar matrix of order n in which the elements of the principal diagonal are the products of elements equidistant from the opposite extremities of the secondary diagonals of the factors. Hence it results that a necessary and sufficient condition that a secondary diagonal matrix whose elements are ordinary complex numbers generates a group of finite order is that the product of every pair of numbers equidistant from the extremities of its secondary diagonal is a root of unity.

A secondary diagonal matrix of order n transforms a quasi-scalar matrix of this order

$$\begin{bmatrix} 0 & 0 & \cdots & b_1 \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & b_{n-1} & \cdots & 0 \\ b_n & 0 & \cdots & 0 \end{bmatrix}^{-1} \cdot \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & a_n \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & \cdots & b_1 \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & b_{n-1} & \cdots & 0 \\ b_n & 0 & \cdots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} a_n & 0 & \cdots & 0 \\ 0 & a_{n-1} & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & a_1 \end{bmatrix}$$

into a quasi-scalar matrix which may be obtained from the original quasi-scalar matrix by interchanging the numbers equidistant from the extremities of its principal diagonal. Moreover, the product of a quasi-scalar matrix of order n and a secondary diagonal matrix of this order is a secondary diagonal matrix of order n. Hence it results that *if a group of finite order is represented by quasi-scalar matrices and by secondary diagonal matrices of a common order this group must contain an abelian subgroup of half its order, which is composed of the operators represented by its quasi-scalar matrices.* In fact, the entire group may be abelian but the given subgroup composed of the quasi-scalar matrices *must* be abelian.

The fact that every abelian group can be represented by quasi-scalar matrices and that every abelian group of even order can be represented also by quasi-scalar and secondary diagonal matrices is evident. It is not quite so evident that every possible non-abelian group G which involves an abelian group H of half its order can be represented by quasi-scalar and secondary diagonal matrices. The case where the order of the H is either odd or twice an odd number is easily treated since in this case the independent generator of this subgroup can be represented by quasi-scalar matrices in such a way that a secondary diagonal matrix, whose square is an arbitrary one of these quasi-

scalar matrices, transforms any arbitrary number of them into themselves and each of the remaining ones into its inverse, and every automorphism of period 2 of such an abelian group can be obtained in this way, as was observed in the preceding section.

Since Sylow subgroups of the same order must correspond to themselves in every possible automorphism of an abelian group it remains only to prove that quasi-scalar matrices representing independent generators of the Sylow subgroup of order 2^m in H can be so chosen that any possible automorphism of order 2 of this subgroup may be secured by the given secondary diagonal matrix. That this can be done results almost directly from the theorems established at the end of the preceding section. Hence the following theorem: *Every possible group which involves an abelian subgroup of half its own order can be represented by square matrices of a common order whose elements are all zero except those in one of the two diagonals, which are ordinary complex numbers.*

ON INFINITE REGIONS*

BY

WILLIAM F. OSGOOD

The object of the present paper is to lay down a general definition of infinite regions which will include the cases of projective geometry, the geometry of inversion, the geometry of the space of analysis, and other geometries which, like these, may be based on a set of elements consisting of the points of ordinary space of n complex dimensions, extended by a complex $(n-1)$-dimensional set of ideal elements—the so-called "points at infinity." This latter manifold is algebraic in character, being in the familiar cases the line or plane at infinity, the null circle or sphere at infinity, or the n hyperplanes at infinity.

We are not concerned directly with geometries based on ordinary real space—real projective geometry, the real geometry of inversion, etc. In these geometries the infinite region is sometimes a manifold of only one less dimension than the space it closes, and the extended space is then not necessarily linearly simply connected. Here, however, the space considered is of $2n$ real dimensions, and it is closed by a manifold of two less dimensions, the resulting extended space being always linearly simply connected.

A prototype for one class of cases included here is given by the geometry of inversion, to which reference has already been made. In that geometry a one-to-one relation is established between the elements of the group of circular transformations, whose transformations operate on the points of complex n-dimensional space, and the group of collineations in space of one higher dimension, which carry a certain quadric surface (or hypersurface, when $n > 2$) over into itself. Furthermore, a one-to-one relation between the points of the n-dimensional space and the points of the quadric is established. Now the quadric is taken in projective space, and it is closed by the ideal points in which it meets the plane at infinity. But ordinary n-dimensional space is not closed, and it is not until it has been closed by a suitable set of ideal elements—the so-called *infinite region*—that such a correspondence is possible. Aside, however, altogether from a correspondence of this kind, it is necessary to introduce the same set of ideal elements if the transformations of the first named group are to carry each point of the set on which they operate into a point of that set.

* Presented to the Society, December, 1915.

The immediate object of the present paper is, however, not geometric, but analytic. Weierstrass stated the theorem that a function of n complex variables which is meromorphic at every point of space is rational; and he understood by *space* the extended space of analysis—the n spheres of the n complex variables. It was for this case that Hurwitz gave the proof. But the theorem is true for other spaces,* and it is the purpose of this paper to show that it holds for all spaces closed by an infinite region according to the definitions which follow.

With the attainment of this end a gap in the theory of functions of several complex variables has been filled and the purpose of this investigation accomplished. But the formulation of the definition here considered prepares the way for the statement of an algebraic-geometrical problem of great generality, namely: To determine all extensions of the ordinary space of n complex variables by an infinite region, of $n-1$ complex dimensions, such that the extended space admits a transformation into itself which is one-to-one and regular at every point; and for a given space to determine all such transformations.† The present paper contributes to this problem the result that all such transformations are birational.

One final remark. The introduction of an infinite region is a *convenience*, not a *necessity*. It is the transformations—in practice, a group of transformations—and invariants with respect to the same that are the essential thing, and the formulation of the theorems can always be so modified as to retain the substance and still preserve accuracy. Thus in the case of Weierstrass's theorem, § 2 below could be omitted altogether. In § 3 we should then demand that the function $\Omega(z_1, \cdots, z_n)$ go over by each of the transformations corresponding to a set of functions (F) belonging to a given class $\mathfrak{F}$ into a new function $\Omega'(z_1', \cdots, z_n')$ meromorphic except perhaps for removable singularities in the neighborhood of a point A'. Such a function is then shown to be rational by the same analysis as that used in § 3.

1. On a certain class of functions

Condition A. Let each of the functions $F_i(z_1', \cdots, z_n')$, $i = 1, \cdots, n$, be meromorphic in the finite point $A' = (a_1', \cdots, a_n') = (a')$, and let at least one of them have a singular point there. Make the transformation

$$z_i = F_i(z_1', \cdots, z_n') \qquad (i = 1, \cdots, n). \tag{1}$$

* A first generalization for the case of projective space was given by the author in these Transactions, vol. 13 (1912), p. 159, and further extensions for the intermediate spaces (cf. *Madison Colloquium*, p. 141) have been obtained by Dr. Jackson, in a forthcoming number of the Journal für Mathematik.

† For projective space and the space of analysis it is readily shown that the totality of such transformations is given by the linear transformations.

If these functions are so constituted that they carry that part σ' of a certain neighborhood s' of A', in which they are all analytic, in a one-to-one manner over into a region σ, then they shall be said to satisfy *Condition A*.

More precisely, they satisfy Condition A *at the point* A'. It is evident that they also satisfy Condition A at every interior point of s' which does not lie in σ'; i. e., in which at least one of the functions has a singularity.

We shall frequently denote a set of n such functions F_i by (F).

A transformation

$$w_i = \omega_i(z_1, \cdots, z_n) \qquad (i = 1, \cdots, n) \tag{2}$$

shall be said to be *regular* at a point $(z) = (a)$ if each of the functions ω_i is analytic at (a) and the jacobian

$$\frac{\partial(\omega_1, \cdots, \omega_n)}{\partial(z_1, \cdots, z_n)}$$

does not vanish at (a). The inverse of (2),

$$z_i = \Omega_i(w_1, \cdots, w_n) \qquad (i = 1, \cdots, n), \tag{3}$$

is then also regular at the point $(w) = (b)$, where

$$b_i = \omega_i(a_1, \cdots, a_n) \qquad (i = 1, \cdots, n).$$

If the transformations (2) and (3) are analytic at the points (a) and (b) respectively, and if one is the inverse of the other, then each is regular. For the product of their jacobians is unity, and hence neither jacobian can vanish. By virtue of a theorem due to Clements,* if the inverse of (2), the functions ω_i being still analytic at (a), is merely single-valued for points (w) which correspond to points (z) in the neighborhood of the point $(z) = (a)$, then (2) is regular.

It is evident that, if the functions $F_i(z_1', \cdots, z_n')$ satisfy Condition A at a point $A' = (a')$, and if we make a regular transformation

$$w_i' = \omega_i'(z_1', \cdots, z_n') \qquad (i = 1, \cdots, n),$$

whereby the point $(z') = (a')$ is carried into the point $(w') = (b') = B'$, the functions F_i going into Φ_i:

$$F_i(z_1', \cdots, z_n') = \Phi_i(w_1', \cdots, w_n') \qquad (i = 1, \cdots, n),$$

then the functions Φ_i will satisfy Condition A at the point B', and the transformation

$$z_i = \Phi_i(w_1', \cdots, w_n') \qquad (i = 1, \cdots, n),$$

will carry a region τ' consisting of those points of a certain neighborhood

* These Transactions, vol. 14 (1913), p. 328.

of B', in which the functions $\Phi_i(w_1', \cdots, w_n')$ are all analytic, into a region τ which lies in σ. The relation between the F_i and the Φ_i is, of course, in this respect reciprocal.

The converse is also true, as we will presently show. First, however, a definition.

Definition. Let (F) and (Φ) be two sets of functions which satisfy Condition A in the points $A' = (a')$ and $B' = (b')$ respectively. Then these two sets are said to be *equivalent* to each other in the points A' and B' if, the region σ' having been chosen arbitrarily small, it is then possible to find a region τ' such that τ lies in σ; and reciprocally, the region τ' having been taken arbitrarily small, there is then a region σ' such that σ lies in τ.

On the other hand, the two sets of functions (F) and (Φ) shall be said to be *distinct* if σ' and τ' can be so chosen that σ and τ have no point in common. This classification is not intended to be exhaustive.

THEOREM. *If $F_i(z_1', \cdots, z_n')$ and $\Phi_i(w_1', \cdots, w_n')$ are two sets of equivalent functions, and if we set*

$$F_i(z_1', \cdots, z_n') = \Phi_i(w_1', \cdots, w_n') \qquad (i = 1, \cdots, n), \tag{4}$$

then these n equations define a regular transformation of the neighborhood of A' on the neighborhood of B'.

In fact, there corresponds to each point (z') of a certain neighborhood σ' of A' through the equations

$$z_i = F_i(z_1', \cdots, z_n') \qquad (i = 1, \cdots, n),$$

a point (z) of σ, and this point lies in τ. As a point of the latter region it leads through the transformation

$$z_i = \Phi_i(w_1', \cdots, w_n') \qquad (i = 1, \cdots, n),$$

to a point (w') of τ'. Since the jacobian is each time different from 0, we thus obtain n functions

$$w_i' = \omega_i(z_1', \cdots, z_n') \qquad (i = 1, \cdots, n), \tag{5}$$

each analytic in each point of σ', and also finite in σ'. These functions can have, therefore, only removable singularities in the neighborhood of A', and consequently they yield functions each analytic at A'.

Since the relation between the functions F_i and Φ_i is reciprocal, we have thus proven that the equations (4) admit a solution with regard to the z_i''s,

$$z_i' = \Omega_i(w_1', \cdots, w_n') \qquad (i = 1, \cdots, n),$$

such that Ω_i is analytic at B', and hence the transformation (5) is regular at A'.

COROLLARY. *If A_1' and B_1' are any two points of suitably chosen neighborhoods of A' and B' respectively, lying on the boundary of σ' and τ', then the above functions (F) and (Φ) are either equivalent or distinct in A_1' and B_1'.*

The functions will be equivalent if A_1' and B_1' correspond to each other by virtue of the transformation (5); otherwise the functions will be distinct.

Condition B. Two sets of functions (F) and (Φ) which satisfy Condition A shall be said to satisfy *Condition B* if they are either equivalent or distinct.

More precisely, they satisfy Condition B *in the points* A' *and* B'. It follows, then, at once that if A_1' and B_1' be arbitrary points of certain neighborhoods of A' and B' respectively, lying on the boundary of σ' and τ', then the functions (F) and (Φ) also satisfy Condition B in the points A_1' and B_1'.

It is clear that when the above functions (F) are considered first at A' and then at A_1', they necessarily yield two sets of functions satisfying Condition B at these two points; for they are distinct.

Condition C. Let m sets of functions $(F^{(k)})$, $k = 1, \cdots, m$, be given which satisfy Condition A respectively in the points $A'^{(k)}$, and let $\sigma'^{(k)}$ denote definite regions σ' corresponding respectively to each of these points. Moreover, let it be possible each time to find a larger region $\bar{\sigma}'^{(k)}$ such that the corresponding region $\bar{s}'^{(k)}$ contains $\sigma'^{(k)}$ wholly in its interior.

Let $A_1'^{(p)}$ and $A_1'^{(q)}$ be any two boundary points of $\bar{\sigma}'^{(p)}$ and $\bar{\sigma}'^{(q)}$ which are interior to the corresponding regions $\bar{s}'^{(p)}$ and $\bar{s}'^{(q)}$. Then $(F^{(p)})$ and $(F^{(q)})$ shall satisfy Condition B in the points $A_1'^{(p)}$, $A_1'^{(q)}$.

The above functions $(F^{(k)})$ shall be said to satisfy *Condition C* if every (finite) point outside a certain hypersphere, i. e., every point $(z_1, \cdots, z_n)$, for which

$$G < |z_1|^2 + \cdots + |z_n|^2,$$

lies in some region $\sigma^{(k)}$.

Functions of the Class $\mathfrak{R}$. We are now in a position to define the functions of the class which we wish to use. Let $(F^{(k)})$, $k = 1, \cdots, m$, be m sets of functions which satisfy Condition C, each set being considered only in the points of its region $\bar{\sigma}'^{(k)}$. Let (Φ) be a set of functions satisfying Condition A in a point B', and let (Φ) be equivalent with some $(F^{(k)})$ in a boundary-point $A_1'^{(k)}$ of $\sigma'^{(k)}$. The totality of such functions (Φ), each considered in a certain neighborhood of its point B', constitutes the *Class of Functions* $\mathfrak{R}$.

It is obvious that the class includes each set of functions $(F^{(k)})$ considered not merely with reference to $A'^{(k)}$, but also with reference to any other point $A_1'^{(k)}$ on the boundary of $\sigma'^{(k)}$, in which the functions $F_i^{(k)}$ are not all analytic.

2. DEFINITION OF AN INFINITE REGION

By *ordinary* complex space we mean the finite space R of algebraic geometry. Its points are given by n complex coördinates, $(z_1, \cdots, z_n)$. It is some-

times desirable to extend this space by the adjunction of ideal points,—the "points at infinity." These points shall be introduced as follows.

By an *infinite region* shall be understood a class of elements $U = \{P\}$ related among themselves and to the points of ordinary space by the following definitions. The elements P shall be referred to as *points at infinity*, or briefly *points*. As a matter of notation we write $P = (A', F)$, $Q = (B', \Phi)$, etc.

Definition i). A point P of U is given by a finite point

$$A' = (a_1', \cdots, a_n') = (a')$$

and a set of functions (F) which belong to an assigned class $\mathfrak{F}$ and satisfy Conditions A in the point A'.

Definition ii). Two points of U,

$$P = (A', F), \qquad Q = (B', \Phi),$$

coincide (or are *identical*, the *same point*) if and only if the two sets of functions (F) and (Φ) are equivalent in the points A' and B'.

Definition iii). The infinite point $P = (A', F)$ *corresponds* to the finite point A' by virtue of the transformation

$$z_i = F_i(z_1', \cdots, z_n') \qquad (i = 1, \cdots, n); \tag{6}$$

and conversely, A' corresponds to P. Similarly, an infinite point P corresponds to a second infinite point Q if both correspond to the same finite point A'. Any two points at infinity can be made to correspond to each other.

Definition iv). By the *neighborhood* of an infinite point $P = (A', F)$ is meant the region σ and such points of U as correspond by (6) to boundary points of σ' which lie in s'.

A point P of U is said to be a *cluster-point* of a set of finite or infinite points if in every neighborhood of P there are points of the set distinct from P.

Two manifolds $\mathfrak{M}_1$ and $\mathfrak{M}_2$ *meet* or *intersect* in a point P of U if P is a cluster-point for points of each set.

This completes the definition of an infinite region, and we turn now to a theorem of fundamental importance.

THEOREM 1. *An arbitrary straight line of ordinary space,*

$$z_i = c_i + \lambda_i t \qquad (i = 1, \cdots, n), \tag{7}$$

$$0 < \sum_{i=1}^{n} |\lambda_i|,$$

where t ranges over the whole finite t-plane, meets an infinite region U in one point P, and in no second point Q distinct from P.

There is obviously at least one region $\sigma^{(k)}$ which contains points of the

line (7) for which, no matter how large the positive number M be chosen, $|t| > M$. The images (z') of these points (z) by the transformation

$$z_i = F_i^{(k)}(z_1', \cdots, z_n') \qquad (i = 1, \cdots, n), \tag{8}$$

lie in $\sigma'^{(k)}$ and have a cluster-point $A_k' = (a_1', \cdots, a_n')$ interior to $\bar{s}'^{(k)}$ (though possibly on the boundary of $s'^{(k)}$). To this point A_k' corresponds by $(F^{(k)})$ a point $P = (A_k', F^{(k)})$ of U in which the line (7) meets U, and thus the first part of the proposition is established.

Let

$$t' = \frac{1}{t}, \qquad F_i^{(k)} = \frac{H_i(z_1', \cdots, z_n')}{G_i(z_1', \cdots, z_n')}, \qquad (i = 1, \cdots, n).$$

On substituting for z_i in (8) its value from (7) we have

$$c_i + \frac{\lambda_i}{t'} = \frac{H_i}{G_i}$$

or

$$\lambda_i G_i(z_1', \cdots, z_n') - t'[H_i(z_1', \cdots, z_n') - c_i G_i(z_1', \cdots, z_n')] = 0. \tag{9}$$

These equations are satisfied, in particular, by an infinite set of values

$$(t'^{(\mu)}, z_1'^{(\mu)}, \cdots, z_n'^{(\mu)}), \qquad (t'^{(\mu)} \neq 0),$$

where

$$\lim_{\mu=\infty} t'^{(\mu)} = 0$$

and the points $(z_1'^{(\mu)}, \cdots, z_n'^{(\mu)})$ have A_k' as a cluster-point.

We have, then, in (9) a system of n equations in the $n+1$ variables t', $z_1', \cdots, z_n'$, to which Weierstrass's second implicit function theorem* applies. From that theorem we infer that the simultaneous solutions of (9) in the neighborhood of the point $(t', z_1', \cdots, z_n') = (0, a_1', \cdots, a_n') = (0, a')$ are given by one or more algebroid equations of the form

$$v_{m+1}^{\nu} + C_1 v_{m+1}^{\nu-1} + \cdots + C_\nu = 0, \tag{10}$$

where $C_q(v_1, \cdots, v_m)$, $q = 1, \cdots, \nu$, is analytic in the point $(v) = (0)$ and vanishes there, and where $v_{m+2}, \cdots, v_{n+1}$ are single-valued functions on the configuration (10); the variables $v_1, \cdots, v_{n+1}$ being linear functions of $t', z_1', \cdots, z_n'$.

But m must equal 1, since otherwise to a given t' near 0 there would correspond more than one point (z') in σ' and hence more than one point (z) in σ.

For the same reason, ν must equal 1, too, and the number of algebroid equations (10) must also reduce to one. Hence t' itself can be taken as the inde-

* Weierstrass, *Werke*, 3, pp. 79–80; *Madison Colloquium*, p. 192.

pendent variable, and the complete system of simultaneous solutions of (9) is given by n equations

$$z_i' = f_i(t') \qquad (i = 1, \cdots, n),$$

where $f_i(t')$ is analytic in the point $t' = 0$ and $f_i(0) = a_i'$.

From this last result we see that, when an arbitrarily restricted neighborhood s of the point $P = (A_k', F^{(k)})$ has been chosen in advance, a value of M can then be assigned such that *all* points (z) of (7), for which $|t| > M$, lie in s (and of course also in σ_k).

The foregoing analysis has not only established the above theorem, but it yields the further result:

THEOREM 2. *If $P = (A', F)$ be the point in which the line* (7) *meets the infinite region U, and if the neighborhood s of P be transformed by* (1) *on the neighborhood s' of the finite point $A' = (a_1', \cdots, a_n')$, then the line* (7) *goes over into a curve which is analytic at (a') and in the neighborhood of this point lies, with the exception of the single point (a') itself, in the region σ'.*

3. THE EXTENSION OF WEIERSTRASS'S THEOREM

Definition. A function $\Omega(z_1, \cdots, z_n)$ shall be said to be *analytic* or *meromorphic* in a point $P = (A', F)$ of the infinite region U if on making the transformation (1) Ω is carried over into a function $\Omega'(z_1', \cdots, z_n')$ which is analytic or meromorphic at the point A'.

THEOREM. *Let ordinary space be closed by an infinite region U given by the above definitions, and let $\Omega(z_1, \cdots, z_n)$ be a function which is meromorphic at every point of the extended space. Then Ω is a rational function.*

The proof can be given by means of the following theorem,* which follows at once from the reasoning Hurwitz employed to prove Weierstrass's theorem in the space of analysis: If $f(z_1, \cdots, z_n)$ is a rational function of each individual variable, when all the others are assigned arbitrary values in the neighborhood of a certain fixed point, O, and if f is analytic in all the variables at O, then f is rational in all its arguments.

Let Ω be analytic at the origin—more specifically, throughout the region

$$|z_i| < h \qquad (i = 1, \cdots, n).$$

Let (c) be a point of this region, and consider the function of z_k alone,

$$\Omega(c_1, \cdots, c_{k-1}, z_k, c_{k+1}, \cdots, c_n). \tag{11}$$

This function is analytic at the point $z_k = 0$. Let $z_k = \alpha$ be the nearest singular point. Then it is readily shown that the function (11) has a pole at α. For, in the neighborhood of the point $(c_1, \cdots, c_{k-1}, \alpha, c_{k+1}, \cdots, c_n)$,

* Cf. *Madison Colloquium*, p. 145.

we have

$$\Omega(z_1, \cdots, z_n) = \frac{h_0 + h_1(z_k - \alpha) + \cdots}{g_0 + g_1(z_k - \alpha) + \cdots},$$

where g_i and h_i are functions of $(z_1, \cdots, z_{k-1}, z_{k+1}, \cdots, z_n)$, each analytic at the point $(c_1, \cdots, c_{k-1}, c_{k+1}, \cdots, c_n)$, and where g_0 vanishes at this point. There must, however, be a g_m that does not vanish there, since otherwise, the fraction being assumed to be in its lowest terms, every point α' of the neighborhood of α would lead to a singular point $(c_1, \cdots, c_{k-1}, \alpha', c_{k+1}, \cdots, c_n)$ of the function $\Omega(z_1, \cdots, z_n)$, and hence it would follow that for *every* value of α' the above point would be singular. But when $\alpha' = 0$, $\Omega(z_1, \cdots, z_n)$ is analytic in the point in question.

Hence it appears that the function (11) has no other singular points except poles in the finite z_k-plane.

Finally, consider the point $P = (A', F)$, in which the line

$$z_i = c_i \qquad (i = 1, \cdots, k-1, k+1, \cdots, n), \tag{12}$$

meets the infinite region U. By hypothesis, Ω goes over by the transformation (1) into a function $\Omega'(z_1', \cdots, z_n')$ meromorphic at A', and by the last theorem of § 2 the line goes over by the same transformation into a curve analytic at A' and lying, except for A', wholly in σ'. Moreover, the function $\Omega'(z_1', \cdots, z_n')$ has no second singular point of the neighborhood of A' in common with this curve, since otherwise every point of the curve in this neighborhood would be a singular point of Ω', and hence every point of (12) would be a singular point of $\Omega(z_1, \cdots, z_n)$. It follows, then, that the function Ω', taken along the curve in question, either remains finite or becomes infinite at A', and hence the function (11) is either analytic at the point $z_k = \infty$ or else has a pole there.

Hence the function (11) is rational in z_k, and all the hypotheses of the foregoing lemma are fulfilled. The function $\Omega(z_1, \cdots, z_n)$ is, therefore, rational in all n arguments.

4. On the transformations of a closed space into itself

Regular Transformations. We have already laid down a definition according to which a transformation (2) is said to be regular at a finite point A, $(z) = (a)$, the transformed point B, $(w) = (b)$, also being finite. The definition shall now be extended to the case that one or both of the points A, B are at infinity.

Let $P = (A', F)$ be a point of U; let B, $(w) = (b)$, be a finite point; and let T be a transformation whereby the points of a certain neighborhood of P are referred in a one-to-one manner to the points of a corresponding neigh-

borhood of B. Then, through the transformation (1), a one-to-one correspondence $\mathfrak{T}$ is established between the points of the above neighborhood of B and the points of a certain neighborhood of A'. And now the transformation T shall be said to be *regular* at P (and its inverse *regular* at B) if $\mathfrak{T}$ is regular, i. e., if $\mathfrak{T}$ can be expressed by equations of the form

$$w_i = \phi_i(z_1', \cdots, z_n') \qquad (i = 1, \cdots, n).$$

where the latter transformation is regular at A' and carries this point into B.

Finally, let $P = (A', F)$ and $Q = (B', \Phi)$ both be points of U, and let T be a transformation whereby the points of a certain neighborhood s of P are referred in a one-to-one manner to the points of a certain corresponding neighborhood t of Q. Then a transformation $\mathfrak{T}$ of a certain neighborhood s' of A' on a certain neighborhood t' of B' is herewith defined. And now T shall be said to be *regular* at P if $\mathfrak{T}$ is regular at A', and B' corresponds to A'. The inverse transformation will then be regular at Q.

It follows at once from the foregoing definitions that if a transformation T is regular at a finite or infinite point P of the (z) space, and is given in the neighborhood of P by the equations

$$w_i = \omega_i(z_1, \cdots z_n),$$

then the functions ω_i are meromorphic at P. If, furthermore, the inverse transformation be represented by

$$z_i = \Omega_i(w_1, \cdots w_n),$$

and if $P = (A', F)$ be a point at infinity, while its image B is finite, then the functions Ω_i satisfy Condition B and

$$P = (B, \Omega).$$

Lastly, if $P = (A', F)$ and $Q = (B', \Phi)$ are both at infinity, then Q can be represented in the form $Q = (A', G)$, where the points (z) and (w) given by the equations

$$z_i = F_i(z_1', \cdots, z_n'), \qquad w_i = G_i(z_1', \cdots, z_n') \quad (i = 1, \cdots, n)$$

are the images of each other under T.

THEOREM 1. *Let ordinary space be closed by an infinite region according to the foregoing definitions. Let T be a transformation of the extended space into itself which is regular at every point. Then T is one-to-one without exception, and is expressed by equations of the form*

$$\text{(I)} \qquad w_i = \omega_i(z_1, \cdots, z_n) \qquad (i = 1, \cdots, n),$$

where ω_i is rational in all n arguments, the inverse transformation

$$(II) \qquad z_i = \Omega_i(w_1, \cdots, w_n), \qquad (i = 1, \cdots, n).$$

also being rational.

Since the extended space is linearly simply connected and closed, it follows that the transformation T must be one-to-one without exception.

It is clear that there exists a pair of finite points A' and B', images the one of the other, at which T is regular:

$$(13) \qquad w_i = \omega_i(z_1, \cdots, z_n) \qquad (i = 1, \cdots, n).$$

Let (Z') be an arbitrary finite point, and let L be a regular curve of finite space connecting A' with (Z'). Then each $\omega_i(z_1, \cdots, z_n)$ can be continued *meromorphically* along L to (Z'). If this were not the case, let (c') be the first point of L encountered in going from A' to (Z'), at which some ω_i cannot be continued further meromorphically. But T is expressed in the neighborhood of (c') by equations of the type (13), the right-hand sides being meromorphic at (c'). Hence we are led to a contradiction.

Thus it appears that each function ω_i in (13) can be continued meromorphically over the entire finite space. Finally, since the transformation T is to be regular in each point of the infinite region U, each function ω_i must be meromorphic in such a point also, and the theorem is established.

COROLLARY. *In no finite point $(z) = (a)$, in which all the functions ω_i are analytic, can the jacobian*

$$\frac{\partial(\omega_1, \cdots, \omega_n)}{\partial(z_1, \cdots, z_n)}$$

vanish.

Moreover, if $P = (A', F)$ be an infinite point and if $\omega_i'(z_1', \cdots, z_n')$ denotes the function into which $\omega_i(z_1, \cdots, z_n)$ is transformed by (1); if finally P is carried by T into a finite point; then $\omega_i'(z_1', \cdots, z_n')$ is analytic at A', and

$$\frac{\partial(\omega_1', \cdots, \omega_n')}{\partial(z_1', \cdots, z_n')}$$

does not vanish at (a').

Lastly, if P is carried by T into a point $Q = (B', \Phi)$ of U, and if

$$w_i' = \overline{\omega}_i(z_1', \cdots, z_n') \qquad (i = 1, \cdots, n)$$

denotes the relation between the points of the neighborhoods of A' and B' corresponding to T, then

$$\frac{\partial(\overline{\omega}_1, \cdots, \overline{\omega}_n)}{\partial(z_1', \cdots, z_n')}$$

does not vanish at A'.

We can proceed now to a converse proposition, which we will state as follows.

THEOREM 2. *Let ordinary space be extended by an infinite region U according to the foregoing definitions, and let T be a transformation of the extended space into itself such that the jacobian relations of the preceding corollary are fulfilled at every point. Then T is regular at every point.*

For, such a closed space is linearly simply connected, and the above jacobian conditions are sufficient that *im Kleinen* the transformation be everywhere regular.

It appears, then, that if ordinary space be extended in a second way by an infinite region U' according to the foregoing definitions, and if T':

$$w_i = \varpi_i(z_1, \cdots, z_n) \qquad (i = 1, \cdots, n),$$

be a transformation of the second space into itself regular at every point, then T', applied to the first space, will yield a transformation one-to-one in general, but with fundamental manifolds.

The question presents itself: Can an arbitrary birational transformation of ordinary space become, on extending space suitably by an infinite region, a transformation of the extended space into itself, which is regular at every point?

A first necessary condition that this be the case is that the jacobian shall not vanish at any finite point at which all the functions $\omega_i(z_1, \cdots, z_n)$ are analytic; and a similar condition must hold for the inverse transformation. It remains to examine the further jacobian conditions corresponding to the foregoing corollary. These conditions bear both on the given transformation and on the particular infinite region introduced.

HARVARD UNIVERSITY

December, 1915

POINT SETS AND ALLIED CREMONA GROUPS*

(PART II)

BY

ARTHUR B. COBLE†

INTRODUCTION

Part I‡ of this account was devoted to a study of the invariants of a set P_n^k of n points in S_k under the group of permutations of the points. The set as a projective figure was mapped by a point P of a space $\Sigma_{k(n-k-2)}$ in which the permutation group appeared as a Cremona group $G_{n!}$. In this part we shall consider the effect upon the set P_n^k of certain Cremona transformations C in S_k. These transformations C are of a special charatcer when $k > 2$ described hereafter by the term *regular*. They are determined essentially by their fundamental points alone and in all important particulars are entirely analogous to the ternary Cremona transformations. These regular Cremona transformations form the *regular Cremona group in* S_k. If one set of fundamental points of a regular transformation C be placed at P_n^k there is determined a new set $P_n'^k$ *congruent to* P_n^k *under* C. The totality of sets $P_n'^k$ congruent in some order to a given set P_n^k is mapped in $\Sigma_{k(n-k-2)}$ by an aggregate of points P' which form a conjugate set under the *extended group* $G_{n,k}$ of P_n^k. This group $G_{n,k}$ in Σ contains $G_{n!}$ as a subgroup and in general is infinite and discontinuous. The major part of this article is devoted to a study of this group.

In § 5 a group $g_{n,k}$ of linear transformations which is isomorphic with $G_{n,k}$ is introduced. The new group brings to light properties both of $G_{n,k}$ and of regular transformations in S_k. An interesting result is a determination of all types of regular transformations with a single symmetrical set of fundamental points. Most of the types are well known but some are novel. In § 6 another group $e_{n,k}$ of linear transformations, also isomorphic with $G_{n,k}$, is defined, which is particularly effective for a discussion of the infinite groups $G_{9,2}$, $G_{8,3}$, and $G_{9,5}$.

The close relation between the associated sets P_n^k and Q_n^{n-k-2} which appeared

* Presented to the Society, April 25, 1914.

† This investigation is being pursued under the auspices of the Carnegie Institution of Washington, D. C.

‡ These Transactions, vol. 16 (1915), p. 155.

throughout Part I is here maintained. These sets have extended groups which are identical while their groups g and e are conjugate under linear transformation.

The first three paragraphs are devoted to the sets P_n^2 partly because the transformations C in S_2 are better known and partly because all the finite types of $G_{n,k}$, viz.: $G_{6,2}$, $G_{7,2} = G_{7,3}$, and $G_{8,2} = G_{8,4}$, occur here. In § 1 a particular order of congruence in S_2 is defined which is applied in § 2 to set forth conditions for congruence in S_2. In § 3 the extended group $G_{n,2}$ is defined and discussed, and the finite cases are identified with important known groups. In § 4 the extended group $G_{n,k}$ is considered.

The invariants of $G_{n,k}$ are taken up in § 8. Only two cases are treated—the $G_{9,2}$ as a sample of the infinite type and the $G_{7,2}$ as a sample of the finite type. The latter case presents a method of attacking the invariants of the ternary quartic which may turn out to be of value.

In this paper no applications of $G_{n,k}$ to the theory of equations are made such as appear for $G_{n!}$ at the end of Part I. The possibilities in this direction will be clear from a later article in which there is presented the important rôle of the extended group $G_{6,2}$ of order 51840 in the problem of determining the lines of a cubic surface.*

* It is not the purpose of this article to develop specific facts concerning definite types of transformations C in S_2. For these, treatises such as Clebsch-Lindemann, *Leçons sur la géométrie*, vol. II, sec. I, chap. IX; Doehlemann, *Geometrische Transformationen*, part II; and Sturm, *Die Lehre von den geometrischen Verwandschaften*, part IV, are available. So far as the author is aware the following points concerning C in S_2 are novel: (*a*) the definition and use in §§1, 2 of a definite mutual order of the F-points of C^{-1} and C; (*b*) the proof in §3 that, for a general point set, congruence does not imply projectivity except in the few particular cases enumerated in §2; (*c*) the development of the group $G_{n,2}$ in §3, of its algebraic relation to the set P_n^2, and of the group $e_{n,2}$ in §6; (*d*) the association of C in S_2 with regular transformations in hyperspace; (*e*) the determination in §6 of all types of C in S_2 with 9 or fewer F-points; and (*f*) the invariants of P_7^2 and P_9^2 under $G_{7,2}$ and $G_{9,2}$ respectively (§8).

In order to present these matters adequately it has been necessary to use, and occasionally convenient to re-prove, some known facts whose origin it is not easy to ascertain. The most original and comprehensive advance in this field is due to S. Kantor in the attempt to determine all types of finite Cremona groups. This work has been perfected by A. Wiman, *Zur Theorie der endlichen Gruppen . . .*, Mathematische Annalen, vol. 48 (1897), p. 195, where full references to the articles of Kantor and others are given. The mappings in §4 from the plane of P_6^2, P_7^2, and P_8^2 to respectively the cubic surface, the quartic curve, and the space sextic of genus four on a quadric cone date back to Clebsch and Noether and constantly reappear in later papers (cf. Wiman, loc. cit.). Articles along these general lines have been published by Snyder, these Transactions, vol. 11 (1910), p. 371 and vol. 12 (1911), p. 354; American Journal of Mathematics, vol. 33 (1911), p. 327.

The notion of a regular transformation in spaces of three or more dimensions appears to be new, as well as the entire discussion of $G_{n,k}$, $g_{n,k}$, and $e_{n,k}$ ($k > 2$) which is based on this notion. But particular regular transformations occur frequently in the literature. Thus for the Geyser involution determined by P_6^3 cf. Sturm, loc. cit.; and for the involution of order 15 determined by P_7^3 cf. Conner, American Journal of Mathematics, vol. 38 (1916).

1. Congruence of point sets in S_2

The general Cremona transformation* C_m of *order* m from the plane E_x to the plane E_y is established by defining a projective correspondence between the lines of E_y and the curves of order m of a net in E_x such that two curves of the net which correspond to lines on a point y have a single variable intersection at x. The remaining $m^2 - 1$ fixed intersections of the two curves arise by requiring that all the curves of the net pass through ρ *fundamental points* or *F-points*, $p_1, \cdots, p_\rho$, with a multiplicity r_i at the point $p_i (i = 1, \cdots, \rho)$. The inverse transformation C_m^{-1} of the same order from E_y to E_x determines a similar net of curves on E_y which pass through the same number ρ of F-points on E_y, $q_1, \cdots, q_\rho$, with the multiplicity s_j at the point q_j $(j = 1, \cdots, \rho)$. The F-points in either plane have no definite correspondents in the other. The directions about the F-point p_i on E_x correspond to the points of a *fundamental curve* or *F-curve* $F_i(y)$ on E_y which is rational, of order r_i, and completely determined by the number α_{ij} of times it passes through the F-points q_j on E_y. From this definition α_{ij} is the number of directions at p_i which correspond to directions at q_j whence α_{ij} is also the number of times the F-curve $F_j(x)$ on E_x which corresponds to q_j on E_y passes through the point p_i. We shall be concerned with the points p_i and q_j only as they may happen to form a part of a general point set P_n^2 or Q_n^2 and therefore will assume hereafter that they are in general position.

By proper conventions it is possible to order the two sets of F-points p, q with respect to each other. Let $p_{a_1}, \cdots, p_{a_g}$ be those points of the set p which have the same multiplicity r; $q_{b_1}, \cdots, q_{b_h}$ those points of the set q which have the same multiplicity s. Then the elements of the matrix $\|\alpha_{ij}\|$ $(i = a_1, \cdots, a_g; j = b_1, \cdots, b_h)$ in general are all alike. There will however be just one particular group of the q's for which $g = h$ and for which also the elements of the square matrix $\|\alpha_{ij}\|$ all have the same value β except for a single element in each line which has the value $\gamma \neq \beta$.† If then for a particular order of $p_{a_1}, \cdots, p_{a_g}$ we arrange the points $q_{b_1}, \cdots, q_{b_g}$ so that the elements γ fall along the principal diagonal of the square matrix a point of the group p corresponds to a definite point of the group q and this correspondence is independent of the particular order in the group p. This convention fails if $g = 2$ or if $g = 1$. If $g = 2$ we require in addition that the elements γ in the principal diagonal be greater than the elements β in the other diagonal. The number σ of groups p for which $g = 1$ is the same as the number of groups q for which $g = 1$. If $p_1, \cdots, p_\sigma$ be the points in these σ groups arranged in order of decreasing multiplicity and if $q_1, \cdots, q_\sigma$ be the similar points similarly

* The properties of this transformation as set forth in Clebsch-Lindemann (loc. cit.) are assumed here.

† Cf. Clebsch-Lindemann, loc. cit., pp. 202–5.

arranged we shall say that p_e and q_e $(e = 1, \cdots, \sigma)$ correspond. By means of the conventions just adopted there is established a definite correspondence between the two sets of F-points of C_m.

We shall say that the point sets P_n^2 and Q_n^2 $(n \geqq 5)$ are *congruent under the Cremona transformation* C_m with ρ F-points if ρ of the pairs p_i, q_i $(i = 1, \cdots, n)$ are corresponding F-points of C_m as defined above and if the remaining $n - \rho \geqq 0$ of the pairs p_i, q_i are pairs of ordinary corresponding points under C_m. Here the requirement $n \geqq \rho$ is an essential part of the definition while the requirement $n \geqq 5$ merely bars the cases $n = 3, 4$ where the definition might apply for $m = 2$ but would not imply a projective property of the point sets. The definition also requires that none of the $n - \rho$ points p or $n - \rho$ points q shall lie on the F-curves of C_m which is of course in line with the hypothesis that P_n^2 or Q_n^2 is a general set.

(1) *The general Cremona transformation* C_m $(m > 2)$ *with* ρ F*-points is projectively determined when there is given the order* m*, the* ρ F*-points in* E_x*, their multiplicities subject to the conditions* $\sum_{i=1}^{i=\rho} r_i^2 = m^2 - 1$, $\sum_{i=1}^{i=\rho} r_i = 3(m-1)$*, and the positions in* E_y *of four corresponding* F*-points.*

The multiplicities of the F-points on E_x determine the net. The F-curves on E_x are determined by the requirement that they have no variable intersection with curves of the net. The orders of the F-curves determine the multiplicities s_j and the numbers α_{ij} so that the particular four F-curves which correspond to the four given points on E_y can be picked out. The residue of each of the four F-curves with regard to the net is a definite pencil of curves so that the four pencils of curves of the net on E_x corresponding to the four line pencils on the given points of E_y are determined. Hence the projective correspondence between the net of curves on E_x and the net of lines on E_y is known and C_m is determined.

(2) *The projective conditions for the congruence of two sets* $P_{\rho+1}^2$ *and* $Q_{\rho+1}^2$ *under* C_m $(m > 2)$ *with* ρ F*-points imply the projective conditions for the congruence of the sets* P_n^2 *and* Q_n^2 $(n > \rho + 1)$ *under* C_m.

For among the pairs of $P_{\rho+1}^2$ and $Q_{\rho+1}^2$ there occur the ρ pairs of corresponding F-points of C_m. If the data of theorem (1) are given and if the further conditions which determine from these data the location of the remaining $\rho - 4$ F-points on E_y have been obtained in some fairly convenient form,* then the conditions for the occurrence of the $(\rho + 1)$-th pair amount to the projective construction of a pair of ordinary corresponding points. This construction can then be applied to the remaining $n - (\rho + 1)$ corresponding pairs. The above remarks apply also to the particular case $m = 2$ if $\rho + 1$ be replaced by $\rho + 2$.

* The form suggested in Clebsch-Lindemann, loc. cit., p. 203, footnote (2), would seldom be practically useful.

It is clear that some sort of expression of the conditions for congruence is indispensable for the utilization in geometry or analysis of the Cremona transformations of higher degree. In § 2 projective statements of these conditions for the cases $n < 9$ are given. In § 3 there is indicated a method for stating these conditions analytically which applies to any case.

If the sets P_n^2 and Q_n^2 are congruent under C_m, the first ρ pairs being F-points of C_m, the arithmetical effect of C_m upon curves in E_x which are related to the set P_n is expressed by a linear transformation $L(C_m)$* which for convenience is given here. Let a curve in E_x of order z_0 have the multiplicity z_i at the point p_i of P_n^2. It is transformed by C_m into a curve in E_y of order z_0' which has the multiplicity z_j' at the point q_j of Q_n^2. Then

$$(3) \qquad L(C_m): \begin{cases} z_0' = mz_0 - \sum_{i=1}^{i=\rho} r_i z_i, \\ z_j' = s_j z_0 - \sum_{i=1}^{i=\rho} \alpha_{ij} z_i, & \begin{pmatrix} j = 1, \cdots, \rho; \\ l = \rho+1, \cdots, n \end{pmatrix}. \\ z_l' = -(-1) z_l, \end{cases}$$

It is worth noting that when the z's and z''s refer to the same coördinate system $L(C_m)$ is unique only by virtue of our conventions in regard to the order of congruence. If these were disregarded L might be any one of $n!$ such transformations.

2. Projective conditions for congruence in S_2

In this paragraph the projective conditions on the two sets of ρ F-points of C_m, and the further conditions on a pair of ordinary corresponding points when $n > \rho$, will be determined for all transformations C_m for which $n < 9$. This is a natural limit since the number of types of C_m is infinite for $\rho \geq 9$. The conditions are given in the form of constructions for the points to be determined, all of which are of rather elementary character except possibly that for the ninth base point of a pencil of cubics.†

The various types of C_m to be considered are furnished by the following table where α_j is the number of F-points of multiplicity j:

	C_2	C_3	C_4	D_4	C_5	D_5	D_6	E_6	E_6^{-1}	D_7	E_7	D_8	E_8	E_8^{-1}	E_8'	E_9	E_9^{-1}	E_9'
α_1	3	4	3	6	0	3	1	4	3	0	2	0	2	0	1	1	0	0
α_2		1	3	0	6	3	4	1	4	3	3	0	0	5	3	1	3	4
α_3				1		1	2	3	0	4	2	7	5	2	2	3	3	0
α_4								–	1		1		1	0	2	3	1	4
α_5														1			1	

* This transformation has been employed by S. Kantor in his crowned memoir of 1883: *Theorie des transformations periodiques univoques:* Naples (De Rubertis), 1891, p. 293.

† Cf. Clebsch-Lindemann, loc. cit., vol. III, p. 451.

	E_{10}	E_{10}^{-1}	E'_{10}	E''_{10}	E_{11}	E'_{11}	E_{12}	E_{12}^{-1}	E'_{12}	E_{13}	E'_{13}	E_{14}	E'_{14}	E_{15}	E_{16}	E_{17}
α_1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
α_2	0	1	2	0	1	0	1	0	0	0	0	0	0	0	0	0
α_3	2	5	2	7	2	4	0	3	2	1	0	1	0	0	0	0
α_4	5	0	3	0	3	3	4	1	4	3	6	0	3	1	0	0
α_5		2	1	0	2	0	3	4	1	3	0	6	3	4	3	0
α_6				1		1			1	1	2	1	2	3	5	8

That the table* is complete and that each type exists can be verified by taking the products of the various types and a quadratic transformation whose three F-points are placed either at F-points of C_m or at F-points and ordinary points if thereby the product has not more than 8 F-points. The notation for the transformations is so chosen that the letter C, D, E indicates 6 or fewer, 7, 8 F-points respectively; the subscript indicates the order; and the superscript indicates an arrangement according to the maximum multiplicity of the F-points. The well-known types C_2, C_5, D_8, and E_{17} contain the F-points symmetrically and will prove especially useful.

In what follows the first of the two subscripts of a point indicates its order in a set and the second indicates its multiplicity as an F-point of C_m. We shall now take up the particular types.

C_2: This transformation requires two congruent sets of four points for its determination. Two sets of five points, $p_{1,1}, p_{2,1}, p_{3,1}, p_{4,0}, p_{5,0}$ and $q_{1,1}, q_{2,1}, q_{3,1}, q_{4,0}, q_{5,0}$ are congruent under C_2 when *the set P_5^2 as written is projective to the set Q_5^2 with the last two points interchanged.* We can infer from this fact that the general transformation C_m can be constructed by projectivities alone. The fact can be verified analytically and extended easily to the transformation $C^{(k)}$ in S_k obtained by inverting the variables and then effecting a projectivity.

C_5: Two sets of seven points $p_{1,2}, \cdots, p_{6,2}, p_{7,0}$ and $q_{1,2}, \cdots, q_{6,2}, q_{7,0}$ are congruent under C_5 when the sets $p_{1,2}, \cdots, p_{6,2}$ and $q_{1,2}, \cdots, q_{6,2}$ are the associated sets whose construction is given in I § 1 (10). Then the sextic line pencil from $p_{7,0}$ to P_6^2 is projective to that from $q_{7,0}$ to Q_6^2.

C_4: If the sets $p_{1,2}, \cdots, p_{3,2}, p_{4,1}, \cdots, p_{6,1}, p_{7,0}$ and $q_{1,1}, \cdots, q_{3,1}, q_{4,2}, \cdots, q_{6,2}, q_{7,0}$ are congruent under C_4, then the product of C_4 and a C_5 with congruent sets $q_{1,2}, \cdots, q_{6,2}, q_{7,0}$ and $r_{1,2}, \cdots, r_{6,2}, r_{7,0}$ is a C_2 with congruent sets $p_{1,0}, \cdots, p_{3,0}, p_{4,1}, \cdots, p_{6,1}, p_{7,0}$ and $r_{1,0}, \cdots, r_{3,0}, r_{4,1}, \cdots, r_{6,1}, r_{7,0}$. Given then the set P_7^2 we construct the auxiliary set R_7^2 congruent to P_7^2 under a C_2 of the above type and then construct the required set Q_7^2 as congruent to R_7^2 under C_5.

* A similar table is given by Kantor, l. c., p. 280 which includes a non-existent type, $m = 11$, $\alpha_3 = 5$, $\alpha_5 = 3$.

C_3: If the F-points of C_3 are $p_{1,2}, p_{2,1}, \cdots, p_{5,1}$ and $q_{1,2}, q_{2,1}, \cdots, q_{5,1}$, the product of C_3 and the C_2 defined by the congruent sets $q_{1,1}, \cdots, q_{3,1}, q_{4,0}, q_{5,0}$ and $r_{1,1}, \cdots, r_{3,1}, r_{4,0}, r_{5,0}$ is a C_2' for which $p_{1,1}, p_{2,0}, p_{3,0}, p_{4,1}, p_{5,1}$ are congruent to $r_{1,1}, r_{3,0}, r_{2,0}, r_{5,1}, r_{4,1}$. This requires *that* $p_1, \cdots, p_5$ *and* $q_1, \cdots, q_5$ *be congruent under a projectivity* π. Since both π and C_3 transform the line $\overline{p_1, p_i}$ into $\overline{q_1, q_i}$ and the conic $\overline{p_i p_j}\ \overline{p_k p_l}$ into $\overline{q_i q_j}\ \overline{q_k q_l}$ $(i, j, k, l = 2, 3, 4, 5)$ their effect on the line pencil on p_1 and on the conic pencil on $p_2, \cdots, p_5$ is the same. If then π transforms $p_{6,0}$ into $\overline{q}_{6,0}$ the correspondent $q_{6,0}$ of $p_{6,0}$ under C_3 is the partner of $\overline{q}_{6,0}$ in the involution cut out on the line $q_{1,2}\,\overline{q}_{6,0}$ by conics on $q_2, \cdots, q_5$.

D_8: Let the F-points of D_8 be $p_{1,3}, \cdots, p_{7,3}$ and $q_{1,3}, \cdots, q_{7,3}$. Let F_{p_i} and F_{q_i} be the F-curves with double points at p_i and q_i respectively. Since the pencil of cubics with a given direction at q_1 and on Q_7^2 meets F_{q_1} in a definite point there is a binary projectivity between directions at q_1 and points on the rational cubic F_{q_1} in which the directions on $F_{q_2}, \cdots, F_{q_7}$ correspond to the points $q_2, \cdots, q_7$. By cutting F_{q_1} by lines on q_1 we find that the points $q_2, \cdots, q_7$ on F_{q_1} are projective to the line pencil $q_1 - q_2, \cdots, q_7$. Hence these six lines are projective to the tangents of $F_{q_2}, \cdots, F_{q_7}$ at q_1. Since lines on p_1 are projective under D_8 to the quintic pencil $q_1, q_2^2, \cdots, q_7^2$ or to its tangent line pencil at q_1 we see that the six lines $p_1 - p_2, \cdots, p_7$ are projective to the tangents at q_1 of $F_{q_2}, \cdots, F_{q_7}$ and therefore to the six lines $q_1 - q_2, \cdots, q_7$. Hence *the sets* P_7^2 *and* Q_7^2 *congruent under* D_8 *are also congruent under a projectivity* π. But D_8 and π have the same effect upon the 21 degenerate cubics of the net on P_7^2 and therefore upon every curve of the net as well. If then $p_{8,0}$ and $q_{8,0}$ correspond under D_8 and if $p_{8,0}$ and $\overline{q}_{8,0}$ correspond under π, $q_{8,0}$ is the ninth base point of the pencil of cubics on Q_7^2 and $\overline{q}_{8,0}$.

In the case of C_4 we used the fact that C_4 and C_2 are paired by multiplication with C_5. We can now use the pairing of D_4, D_5, D_6, D_7 with C_5, C_4, C_3, C_2 respectively by multiplication with D_8 in order to obtain the conditions for congruence under these new transformations. This process is easily followed out and only the results are given below.

D_4: If the sets $p_{1,3}, p_{2,1}, \cdots, p_{7,1}, p_{8,0}$ and $q_{1,3}, q_{2,1}, \cdots, q_{7,1}, q_{8,0}$ are congruent under D_4 then the sets $p_2, \cdots, p_7$ and $q_2, \cdots, q_7$ are associated and $p_{1,3}, q_{1,3}$ correspond under the C_5 determined by the associated sets. If also $p_{8,0}, \overline{q}_{8,0}$ correspond under this C_5 then $q_{8,0}$ is the ninth base point of the pencil of cubics on $q_1, \cdots, q_7, \overline{q}_{8,0}$.

D_5: If the sets $p_{1,1}, \cdots, p_{3,1}, p_{4,2}, \cdots, p_{6,2}, p_{7,3}, p_{8,0}$ and $q_{1,2}, \cdots, q_{3,2}, q_{4,1}, \cdots, q_{6,1}, q_{7,3}, q_{8,0}$ are congruent under D_5 then the sets $p_{1,2}, \cdots, p_{3,2}, p_{4,1}, \cdots, p_{6,1}, p_{7,0}$ and $q_{1,1}, \cdots, q_{3,1}, q_{4,2}, \cdots, q_{6,2}, q_{0,7}$ are congruent under C_4. If the further ordinary pair $p_{8,0}, \overline{q}_{8,0}$ correspond under this C_4' then $q_{8,0}$ is the ninth base point of the pencil of cubics on $q_1, \cdots, q_7, \overline{q}_{8,0}$.

D_6: If the sets $p_{1,1}, p_{2,2}, \cdots, p_{5,2}, p_{6,3}, p_{7,3}, p_{8,0}$ and $q_{1,1}, q_{2,2}, \cdots, q_{5,2}, q_{6,3}, q_{7,3}, q_{8,0}$ are congruent under D_6 then the sets $p_{1,2}, p_{2,1}, \cdots, p_{5,1}, p_{6,0}, p_{7,0}, p_{8,0}$ and $q_{1,2}, q_{2,1}, \cdots, q_{5,1}, q_{6,0}, q_{7,0}, \overline{q}_{8,0}$ are congruent under C_3 and $q_{8,0}$ is the ninth base point of the pencil on $q_1, \cdots, q_7, \overline{q}_{8,0}$.

D_7: If the sets $p_{1,2}, \cdots, p_{3,2}, p_{4,2}, \cdots, p_{7,3}, p_{8,0}$ and $q_{1,2}, \cdots, q_{3,2}, q_{4,3}, \cdots, q_{7,3}, q_{8,0}$ are congruent under D_7 then the sets $p_{1,1}, \cdots, p_{3,1}, p_{4,0}, \cdots, p_{7,0}, p_{8,0}$ and $q_{1,1}, \cdots, q_{3,1}, q_{4,0}, \cdots, q_{7,0}, q_{8,0}$ are congruent under C_2 and $q_{8,0}$ is the ninth base point of the pencil of cubics on $q_1, \cdots, q_7, \overline{q}_{8,0}$.

There remain only the types E with 8 F-points. Since we seek the conditions for congruence of sets with at most 8 points the construction for ordinary pairs need not be considered further. As we shall prove later the sets P_8^2 and Q_8^2 for the type E_{17} are projective. If then $EE_{17} = C_m$ the conditions for congruence of 8 points under E and C_m are the same. In this way E_{17} pairs the transformations as follows: E_{16}, C_2; E_{15}, C_3; E_{14}, D_4; E'_{14}, C_4; E_{13}, D_5; E'_{13}, C_5; E_{12} and E_{12}^{-1}, E_6 and E_6^{-1}; E'_{12}, D_6; E_{11}, E_7; E'_{11}, D_7; E_{10} and E_{10}^{-1}, E_8 and E_8^{-1}, E_{10}, E'_8; $E''_{10}; D_8$; while E'_9, and E_9, E_9^{-1} are paired with themselves. Hence to complete the discussion we have yet to consider only the types E_6, E_6^{-1}, E_7, E_8, E_8^{-1}, E'_8, E_9, E_9^{-1}, E'_9, and E_{17}.

E_6, E_6^{-1}: Since the product $E_6 D_8$ for proper choice of the F-points of D_8 is D_6 we have the following construction for the two sets of F-points of E_6, E_6^{-1}. If in the sets P_8^2 and Q_8^2 congruent under D_6 as above we replace q_1 by $\overline{q}_1$, the ninth base point of the pencil of cubics on Q_8^2 we have sets $p_{1,2}$, $p_{2,1}, \cdots, p_{5,1}, p_{6,3}, \cdots, p_{8,3}$ and $q_{8,4}, q_{2,2}, \cdots, q_{6,2}, q_{6,1}, q_{7,1}, \overline{q}_{1,1}$ congruent in this order under E_6, E_6^{-1}.

E_7: The product $D_5 D_8$ for isolated $q_{6,1}$ in the sets congruent as above under D_5 is E_7. If then we replace q_6 by $\overline{q}_6$, the ninth base point of the pencil on Q_8^2 we have sets $p_{1,2}, \cdots, p_{3,2}, p_{4,1}, p_{5,1}, p_{6,4}, p_{7,3}, p_{8,3}$ and $q_{1,2}, \cdots, q_{3,2}, q_{5,3}, q_{4,3}, q_{8,4}, \overline{q}_{6,1}, q_{7,1}$ congruent in this order under E_7.

E'_8: This type can be obtained from the product $C_4 D_8$. If in the sets congruent as above under C_4 and amplified by the pair $p_{8,0}, q_{8,0}$ we replace q_1 by $\overline{q}_1$ the ninth base point of a pencil on $q_1, \cdots, q_8$ we obtain the sets $p_{1,1}, p_{2,4}, p_{3,4}, p_{4,2}, \cdots, p_{6,2}, p_{7,3}, p_{8,3}$ and $\overline{q}_{1,1}, q_{2,3}, q_{3,3}, q_{4,2}, \cdots, q_{6,2}, q_{7,4}, q_{8,4}$ congruent in this order under E'_8.

E_8, E_8^{-1}: From the product $D_4 D_8$ we find that if in the sets congruent as above under D_4 we replace q_2 by $\overline{q}_2$ the ninth base point of the pencil on Q_8^2 we obtain the set of F-points of E_8^{-1}, viz., $p_{1,3}, p_{2,5}, p_{3,2}, \cdots, p_{7,2}, p_{8,3}$, which is congruent to the set of F-points of E_8, viz., $\overline{q}_{2,1}, q_{8,4}, q_{3,3}, \cdots, q_{7,3}, q_{1,1}$.

E'_9: From a proper product $C_5 C'_5$ we find that, for the sets congruent as above under C_5 and amplified by a further pair $p_{8,0}, q_{8,0}$, if we construct a

set $\overline{q}_{1,0}, \overline{q}_{2,0}, \overline{q}_{3,2}, \cdots, \overline{q}_{8,2}$ congruent under C_5' to $q_{1,0}, q_{2,0}, q_{3,2}, \cdots, q_{8,2}$, then the sets $p_{1,2}, p_{2,2}, p_{3,4}, \cdots, p_{6,4}, p_{7,2}, p_{8,2}$ and $\overline{q}_{2,2}, \overline{q}_{1,2}, \overline{q}_{3,4}, \cdots, \overline{q}_{6,4}, \overline{q}_{8,2}, \overline{q}_{7,2}$ are congruent in this order under E_9'.

E_9, E_9^{-1}: If the sets $p_{1,1}, \cdots, p_{4,1}, p_{5,2}, p_{6,0}, \cdots, p_{8,0}$ and $q_{1,1}, \cdots, q_{4,1}, q_{5,2}, q_{6,0}, \cdots, q_{8,0}$ are congruent under C_3, and if $\overline{q_1}$ is the ninth base point of the pencil on Q_8^2 then the set $p_{1,5}, p_{2,2}, \cdots, p_{4,2}, p_{5,4}, p_{6,3}, \cdots, p_{8,3}$ of F-points of E_9^{-1} is congruent to the set $q_{5,2}, q_{2,3}, \cdots, q_{4,3}, \overline{q}_{1,1}, q_{6,4}, \cdots, q_{8,4}$ of F-points of E_9.

E_{17}: The product of E_{10}'' with congruent sets $p_{1,6}, p_{2,3}, \cdots, p_{8,3}$ and $q_{1,6}, q_{2,3}, \cdots, q_{8,3}$ by a D_8 with congruent sets $q_{1,3}, \cdots, q_{7,3}, q_{8,0}$ and $r_{1,3}, \cdots, r_{7,3}, r_{8,0}$ is a $\overline{D}_8$ with congruent sets $p_{1,3}, p_{2,3}, \cdots, p_{7,3}, p_{8,0}$ and $r_{8,3}, r_{2,3}, \cdots, r_{7,3}, r_{1,0}$, whence the sets $p_2, \cdots, p_7$ and $q_2, \cdots, q_7$ are projective since each is projective to $r_2, \cdots, r_7$. The product of this E_{10}'' by an E_{17} with congruent sets $q_{1,6}, \cdots, q_{8,6}$ and $s_{1,6}, \cdots, s_{8,6}$ is another D_8 with congruent sets $p_{1,0}, p_{2,3}, \cdots, p_{7,3}, p_{8,3}$ and $s_{1,0}, s_{2,3}, \cdots, s_{7,3}, s_{8,3}$. Hence $s_2, \cdots, s_7$ are projective to $p_2, \cdots, p_7$ and therefore to $q_2, \cdots, q_7$. These being any sets of six formed from the sets of 8 F-points of E_{17} there follows that *the sets P_8^2 and Q_8^2 congruent under E_{17} are projective.*

Of all the cases investigated above the only ones for which congruence implies projectivity are (1) that of 5 points under C_2 and C_3, (2) that of 7 points under D_8, and (3) that of 8 points under E_{17}. The order of congruence and of projectivity is the same in all these cases except that of 5 points under C_2. It is important to note however that the projectivity no longer exists when the congruence is extended to include additional pairs.

3. The extended group $G_{n,2}$

In § 1 we have defined the congruence of sets P_n^2 and Q_n^2 under the Cremona transformation C_m. According to (1) if the type of C_m and the location of its F-points among the points of P_n^2 are given, then the set Q_n^2 is projectively determined, i. e., if four of the points of Q_n^2 are fixed at a base the coördinates of the remaining points can be rationally expressed in terms of the coördinates of P_n^2. Of course the same fact is true if the sets be interchanged.

In I, § 6, we have mapped the sets P_n^2 upon the points P of a space $\Sigma_{2(n-4)}$. If we take the sets P_n^2 and Q_n^2 in the canonical form

$$P_n^2: \begin{matrix} 1, & 0, & 0, \\ 0, & 1, & 0, \\ 0, & 0, & 1, \\ 1, & 1, & 1, \\ x_i, & y_i, & u, \end{matrix} \quad (i = 1, \cdots, n-4); \qquad Q_n^2: \begin{matrix} 1, & 0, & 0, \\ 0, & 1, & 0, \\ 0, & 0, & 1, \\ 1, & 1, & 1, \\ x_j', & y_j', & u', \end{matrix} \quad (j = 1, \cdots, n-4);$$

then the coördinates of P and Q, the maps in $\Sigma_{2(n-4)}$ of P_n^2 and Q_n^2 respectively, are x_i, y_i, u and x_j', y_j', u'. Then the above statement can be expressed more briefly as follows:

(4) *If the sets P_n^2 and Q_n^2 are congruent under Cremona transformation their maps P and Q in $\Sigma_{2(n-4)}$ are corresponding points under a Cremona transformation in Σ.*

If P_n^2 is congruent in some order to Q_n^2 under C_m and Q_n^2 is congruent in some order to R_n^2 under C_μ then P_n^2 is congruent in some order to R_n^2 under the product $C_m \cdot C_\mu$ properly taken.* For the $\rho' \leqq n$ F-points of C_μ can all be placed at F-points of C_m^{-1} or at ordinary points of C_m^{-1} which correspond to points of P_n^2. Then the F-points of the product arise either from F-points of C_m or from ordinary points of C_m whose correspondents are F-points of C_μ and in either case are found in P_n^2. Similarly the F-points of the inverse product are found in R_n^2. If an F-curve of C_m is an F-curve of C_μ^{-1} some F-point of P_n^2 and some F-point of R_n^2 form an ordinary pair of the product. Ordinary pairs of both C_m and C_μ take care of themselves. Hence

(5) *Two point sets each congruent in some order to a third are congruent in some order to each other. The aggregate of sets Q_n^2 congruent in some order with a given set P_n^2 is mapped in $\Sigma_{2(n-4)}$ by an aggregate of points Q which form a conjugate set under the thus defined* EXTENDED GROUP *$G_{n,2}$ in $\Sigma_{2(n-4)}$ associated with the point set P_n^2.*

Since the general Cremona transformation in S_2 can be expressed as a properly arranged product of quadratic factors let us determine the effect upon the set P_n^2 given above of a quadratic transformation for which the first four points of P_n^2 are congruent to the first four points of Q_n^2. This transformation is $x' = 1/x, y' = 1/y, u' = 1/u$. Its effect upon the further points of P_n^2 is to invert their coördinates whence the corresponding transformation A *in* $\Sigma_{2(n-4)}$ is obtained by inverting the variables. Since any arrangement in a product can be secured by proper permutation of the points and any quadratic transformation can be obtained by applying such permutations to A we see that

(6) *The extended group $G_{n,2}$ of P_n^2 in $\Sigma_{2(n-4)}$ can be generated by the $G_{n!}$ of P_n^2 (whose generators are given in I, § 7 (64)) and the involutory transformation A which is obtained by inverting the variables in Σ.*

Thus for a general Cremona transformation C_m in S_2 with ρ F-points there will exist an operation of $G_{\rho,2}$ compounded from the above generators which will express the coördinates of the ρ F-points of C_m^{-1} rationally in terms of those of C_m and the same compound formed in $G_{\rho+1,2}$ will give the transformation itself. However the coördinate system in both planes is therewith specially selected. Conversely for every operation of $G_{n,2}$ there will exist for given

* The product is properly taken if the F-points of C_μ are included in the set Q_n^2.

P_n^2 a projectively defined set Q_n^2 which is in some order congruent to P_n^2 as defined in § 1.

The order of $G_{n,2}$ is the number of points Q conjugate to a general point P, i. e., the number of sets Q_n^2 congruent in any one of $n!$ arrangements with P_n^2 under any Cremona transformation with $\rho \leqq n$ F-points provided that no two congruent sets are projective in any order. This proviso is disposed of by the results of § 2 and by the theorem:

(7) *If a point set is congruent to a given* GENERAL *point set under Cremona transformation the two sets cannot be projective in any order if the number of points is nine or more.*

Let us prove that P_n^2, congruent to Q_n^2 under C_m, cannot be projective to Q_n^2 under a collineation K even in the special case when P_n^2 is on a cubic curve E^3 but otherwise unrestricted. We shall see in § 5 that when $\rho \geqq 9$ there exist no C_m's whose F-points all have the same multiplicity. Even when $\rho < 9$ and $n \geqq 9$ ordinary points of multiplicity zero occur so that when $n \geqq 9$ we can assume that points q_j and q_k occur in Q_n^2 whose multiplicities s_j and s_k under C_m are different. Let P_n^2 on E^3 have elliptic parameters $u_1, \cdots, u_n$ where $u + u' + u'' \equiv 0$ is the condition for three points of a line. Then C_m transforms E^3 into a cubic $E^{3'}$ on Q_n^2 in such a way that the point u of E^3 goes into the point u of $E^{3'}$. If K^{-1} transforms Q_n^2 into P_n^2 it must transform $E^{3'}$ back into E^3 so that the point u of $E^{3'}$ goes into the point u' of E^3. Since E^3 is thereby birationally transformed into itself, $u = \pm u' + b$. Now the points q_j and q_k as points u of $E^{3'}$ correspond to the meets outside P_n^2 of their F-curves with E^3, i. e., to $-\sum_{l=1}^{l=\rho} \alpha_{l,j}\, u_l$ and $-\sum_{l=1}^{l=\rho} \alpha_{l,k}\, u_l$; and as points of Q_n^2 they correspond under K^{-1} to points $p_{j'}$ and $p_{k'}$ of P_n^2 whence

$$-\sum_{l=1}^{l=\rho} \alpha_{l,j} = \pm u_{j'} + b, \qquad -\sum_{l=1}^{l=\rho} \alpha_{l,k}\, u_l = \pm u_{k'} + b.$$

Hence $\sum_{l=1}^{l=\rho} (\alpha_{l,j} - \alpha_{l,k})\, u_l = \pm (u_{k'} - u_{j'})$. This relation on the parameters $u_1, \cdots, u_n$ with integral coefficients does not vanish identically for if $u_1 = u_2 = \cdots = u_n = 1$ it takes the value $3(s_j - s_k) \neq 0$. Hence the assumption that K exists requires that P_n^2 be a particular set on E^3 which proves the theorem for a general set on E^3 and *a fortiori* for a general set P_n^2.

We find in § 6 that the number of distinct types of Cremona transformations with 9 F-points is infinite.* Since the number of possible types of congruence is infinite and no types lead to projectively equivalent sets, the number of points Q conjugate to a given point P under $G_{n,2}$ is infinite when $n \geqq 9$. The types of C_m depend upon integer values and the arrangement in Q_n^2 upon a finite number of permutations so that the operations of $G_{n,2}$ form a discontinuous aggregate.

If $n < 9$, $G_{n,2}$ is finite and its order is $n!$ times the number of ways the set

* Kantor, loc. cit., p. 273, Theorem IX.

P_n^2 can be congruent to Q_n^2 under C_m. When $n = 6$, we can use C_1, a collineation, C_2, C_3, C_4, and C_5 in $\binom{6}{0}$, $\binom{6}{3}$, $\binom{6}{4}\binom{2}{1}$, $\binom{6}{2}$, and $\binom{6}{0}$ ways respectively whence the order of $G_{6,2}$ is $6!\,72$. When $n = 7$, we can use C_1, C_2, C_3, C_4, C_5, D_4, D_5, D_6, D_7, D_8 in $\binom{7}{0}$, $\binom{7}{3}$, $\binom{7}{4}\binom{3}{1}$, $\binom{7}{3}\binom{4}{3}$, $\binom{7}{6}$, $\binom{7}{6}$, $\binom{7}{3}\binom{4}{3}$, $\binom{7}{1}\binom{6}{4}$, $\binom{7}{3}$, $\binom{7}{0}$ ways respectively or 2.288 ways in all. Since P_7^2 and Q_7^2 congruent under D_8 are projective we find only 288 projectively distinct types of congruence whence $G_{7,2}$ is of order $7!\,288$. Similarly when $n = 8$ we find by using in all possible ways the transformations of the table in § 2 that the number of ways of congruence is 2.8640 which again give rise to only 8640 projectively distinct ways due to the property of E_{17} noted in § 2.

(8) *The extended group $G_{n,2}$ in $\Sigma_{2(n-4)}$ is infinite and discontinuous if $n \geqq 9$. If $n = 6, 7, 8$ the order of $G_{n,2}$ is $6!\,72$, $7!\,288$, $8!\,8640$ respectively.*

From the form of the generators of $G_{n!}$ and of A we see that if only the generators of $G_{(n-1)!}$ be retained and if A be applied to only the first $n-1$ points of P_n^2, a subgroup $g_{n-1,2}$ of $G_{n,2}$ is obtained which is isomorphic with the $G_{n-1,2}$ derived from P_{n-1}^2. Hence

(9) *The $G_{n,2}$ contains subgroups isomorphic with $G_{n',2}$ where $n' < n$.*

In general this isomorphism is simple and its existence can be seen from the fact that the subgroup $g_{n',2}$ is precisely the $G_{n',2}$ of a set $P_{n'}$ in the $\Sigma_{2(n'-4)}$ lying in $\Sigma_{2(n-4)}$ and determined by equating to zero the first two coördinates of $p_{n'+1}, \cdots, p_n$. An exceptional case occurs when $n' = 5$. Let us take for convenience $n = 6$. The extended group $G_{5,2}$ in Σ_2 is merely a $G_{5!}$. For even though there are 16 ways under which two sets P_5^2 and Q_5^2 can be congruent by using the C_1, the $\binom{5}{3}$ C_2's, and the $\binom{5}{1}$ C_3's these modes of congruence do not lead to projectively distinct sets, e. g., if P_5^2 and Q_5^2 are congruent under A they are projective in the order (45). But if they be enlarged to sets P_6^2 and Q_6^2 congruent under A by adding a pair of ordinary corresponding points the sets are no longer projective in any order. Thus $G_{6,2}$ contains subgroups $g_{5,2}$ of order $5!\,16$ which are in 16 to 1 isomorphism with $G_{5,2}$ of order $5!$ Other cases of exception occur in dropping from $G_{9,2}$ to $G_{8,2}$ and from $G_{8,2}$ to $G_{7,2}$. Here the isomorphism is 2 to 1.

(10) *The index of the subgroup $G_{n-1,2}$ of $G_{n,2}$ is in general the number of rational curves that are determined by points of P_n^2 increased by the number of points.*

For a particular group $G_{n-1,2}$ is determined by isolating a point of P_n^2. By applying the transpositions of $G_{n!}$ this point is transformed into any point. By applying A and then the transpositions there is obtained the set of lines on two of the points. From these in turn there is obtained in the same way the set of conics on five of the points, etc. An exception occurs when $n = 7$ or $n = 8$ for then the rational curves on P_n^2 are paired under the transformations D_8 or E_{17} which leads to the identity in $G_{n,2}$. Here the index is half the usual number.

(11) *The index of each group of the series* $g_{5,2}$, $G_{6,2}$, $G_{7,2}$, $G_{8,2}$, $G_{9,2}$, $\cdots$, $G_{n,2}$ *considered as a subgroup of the following one is* 27, 28, 120, ∞, $\cdots$, ∞ *respectively.*

In order to identify the finite groups $G_{6,2}$, $G_{7,2}$, $G_{8,2}$ with certain known groups let us begin with $G_{7,2}$. For $i, j, k = 1, \cdots, 7$ let I_{0i} denote that involution of $G_{7,2}$ determined by C_5 with F-points at the points of P_7 other than p_i; let I_{ik} denote that involution of $G_{7,2}$ determined by interchanging p_i and p_k; and let I_{0ijk} denote that involution of $G_{7,2}$ determined by C_2 with F-points at p_i, p_j, p_k. These 63 involutions form a conjugate set of generators of $G_{7,2}$. For they include the generators of (6) and these generators permute the 63 involutions transitively. Indeed under the transpositions the sets I_{0i}; I_{ik}; I_{0ijk} are separately permuted while the element $A = I_{0123}$ leaves I_{12}, I_{45}, I_{01}, I_{0145} unaltered and interchanges the pairs (I_{14}, I_{0234}), (I_{0456}, I_{07}). Let us further denote the pairs of rational curves on P_7^2 as follows: $Q_{0i} \equiv \overline{p_i^0\, p_i^2\, p_j\, p_k \cdots}^3$; $Q_{ij} \equiv \overline{p_i\, p_j}^1\, \overline{p_k\, p_l \cdots}^2$. Then it is easy to verify that for $i, j, k \cdots = 0, 1, \cdots, 7$ the involutions permute the pairs of curves as follows:

$$I_{ij} \;:\; (Q_{ij})\,(Q_{ik}, Q_{jk})\,(Q_{kl});$$

$$I_{ijkl}\!:\; (Q_{ij}, Q_{kl})\,(Q_{lm})\,(Q_{mn}, Q_{op}).$$

If we use the base notation as set forth in F. G., II,* for the finite geometry associated with the odd and even theta functions in $p = 3$ variables we find in S_{2p-1} a set of 63 points $P_{ij} = P_{klmnop}$, $P_{ijkl} = P_{mnop}$, each of which determines an involution I_{ij}, I_{ijkl} and a set of 28 O quadrics Q_{ij}. These involutions generate the group of the null system in S_{2p-1}. According to F. G., II (10), they permute the O quadrics precisely as the involutions above permute the pairs of curves. Since the orders of the two groups are the same, $G_{7,2}$ is simply isomorphic with the group of the null system or the group of the double tangents of a quartic.

This last connection can be established by mapping the plane E_x of P_7^2 upon a plane E_y by cubics on P_7^2. A point y of E_y corresponds to two points on E_x. These coincide along a sextic curve with double points at P_7^2 which maps into a quartic curve C^4 on E_y. The 28 degenerate cubics map into the 28 double tangents of C^4 in such a way that the seven which correspond to Q_{0i} form an Aronhold system. The operations of $G_{7,2}$ transform P_7^2 into Q_7^2 and transform the net of cubics and its degenerate curves on P_7^2 into the net and degenerate curves on Q_7^2 but of course they leave C^4 unaltered. Thus to

* Two earlier papers of the writer are cited from time to time: *Finite geometry and theta functions*, these Transactions, vol. 14 (1913), p. 241; and *An isomorphism between theta characteristics and the* $(2p+2)$*-point*, Annals of Mathematics (1916). They are referred to as F. G., I and F. G., II respectively.

the 7!288 operations of $G_{7,2}$ there correspond the 7! ways in which any one of the 288 Aronhold systems may be arranged.

The subgroup of $G_{7,2}$ which leaves Q_{07} unaltered is $G_{6,2}$. It is generated by the 36 involutions I_{ik}, I_{07}, I_{0ijk} $(i, j, k = 1, \cdots, 6)$, and being of index 28 under $G_{7,2}$ has the order 51840. That it is isomorphic with the group of the 27 lines on a cubic surface $C^3(y)$ is evident either from the fact that cubic curves on P_6^2 map E_x upon $C^3(y)$ with isolated lines or that the quartic tangent cone from a point y (the map of p_7) has for double tangent planes the tangent plane at y and the 27 planes from y to the 27 lines of $C^3(y)$. It is clear from the sample I_{07} that the 36 conjugate generators are those which effect the interchange of opposite lines of the 36 double sixes of $C^3(y)$.

In the case of $G_{8,2}$ for subscripts $i, j, k, \cdots = 1, \cdots, 8$ let us denote by I_{ij}, I_{0ijk}, I_{0ij9}, and I_{i9} respectively those involutions of $G_{8,2}$ determined by the transposition $(p_i\, p_j)$, the C_2 with F-points at p_i, p_j, p_k, the C_5 with ordinary points at p_i, p_j, and the D_8 with ordinary point at p_i. It can be shown as above that these 120 involutions constitute a conjugate set of generators of $G_{8,2}$. Let us denote the 120 pairs of rational curves on P_8^2 as follows:

$$Q_{0i9} \equiv \overline{p_i}^{\,0}\,\overline{p_i^3\, p_j^2\, p_k^2 \cdots}^{\,6}; \qquad Q_{ij9} \equiv \overline{p_i\, p_j}^{\,1}\,\overline{p_i\, p_j\, p_k^2\, p_l^2 \cdots}^{\,5};$$

$$Q_{ijk} \equiv \overline{p_l\, p_m \cdots}^{\,2}\,\overline{p_i^2\, p_j^2\, p_k^2\, p_l\, p_m \cdots}^{\,4}; \qquad Q_{0ij} \equiv \overline{p_i^2\, p_k\, p_l \cdots}^{\,3}\,\overline{p_j^2\, p_k\, p_l \cdots}^{\,3}.$$

On the other hand in the finite geometry associated with the theta functions for $p = 4$ there is, for subscripts $i, j, k, \cdots = 0, 1, \cdots, 9$, a set of 126 E quadrics like $Q_{ijklm} = Q_{nopq}$, and a set of 10 E quadrics like Q_i. Also there are 120 O quadrics like Q_{ijk}, 45 points like P_{ij} and 210 points like P_{ijkl}. If we isolate a definite E quadric like Q_0 we find that the group which leaves this quadric unaltered is generated by a conjugate set of 120 involutions determined by the 120 points outside the quadric. According to F. G., II (8), these points comprise the 36 of type P_{ik} and the 84 of type P_{0ijk} $(i, j, k = 1, \cdots, 9)$. Moreover these involutions permute the O quadrics precisely as the 120 involutions above permute the pairs of curves on P_8^2. Since the orders are the same, the $G_{8,2}$ and the group of Q_0 in the finite geometry are simply isomorphic.

This connection with theta functions of genus 4 can also be established by mapping the plane E_x upon a quadric cone $Q^2(y)$ in S_3 by sextic curves with double points at P_8^2.* To a point x there corresponds one point y, to a point y there corresponds two points x which coincide along a 9-ic curve in E_x with triple points at P_8^2. Cubics on P_8^2 map into the generators of Q^2 and the 9-ic curve maps into a sextic of genus 4 on this cone. The 120 degenerate

* In regard to these maps cf. Wiman, *Zur Theorie . . . birationalen Transformationen in der Ebene*, Mathematische Annalen, vol. 48, p. 195.

sextics map into the 120 tritangent planes of this sextic and are therefore associated with the odd theta functions of genus 4. Since P_8^2 has only 8 absolute constants the moduli of the functions are subject to two conditions, one of which is that the functions are Riemannian and the other that the corresponding normal curve is on a nodal quadric. Hence

(12) *$G_{6,2}$ is isomorphic with the group of the lines on a cubic surface, $G_{7,2}$ with the group of the double tangents of a plane quartic curve, and $G_{8,2}$ with the group of the tritangent planes of a space sextic of genus 4 on a quadric cone.*

These three Cremona groups are defined respectively in Σ_4, Σ_6, and Σ_8. Though collineation groups isomorphic with them occur in spaces of lower dimension yet the Cremona groups have a certain initial advantage in that they are in direct algebraic relation with the corresponding geometrical configuration. This advantage appears in § 8 in connection with the invariants of the quartic.

The $G_{n,2}$ is a special case of the extended group $G_{n,k}$ of the set P_n^k in S_k which is defined in the next paragraph and some of its properties appear hereafter from this point of view.

4. Congruence of sets P_n^k in S_k. The extended group $G_{n,k}$ of P_n^k

The theorem that the general Cremona transformation in S_2 can be expressed as a product of quadratic transformations has no analogue for spaces of greater dimension. The possible existence of F-curves, F-surfaces, etc., for the general Cremona transformation C_m^k of order m in S_k so complicates the theory that very few of the relatively simple properties of C_m^2 can be extended to C_m^k. There is however a class of C_m^k's which has properties entirely analogous to C_m^2, namely the class of regular transformations defined below.

The general quadratic transformation in S_2 can be regarded as the product of the quadratic involution A, $x_1' = 1/x_1$, $x_2' = 1/x_2$, $u' = 1/u$, and a projectivity. When projective properties of the transformation are under consideration the projectivity is an unessential factor. So in S_k we shall build up transformations with involutions A of order k of the type $x_i' = 1/x_i$, $u = 1/u$ $(i = 1, \cdots, k)$. Such an involution A is uniquely determined by its $k+1$ F-points and any corresponding pair in general position and we shall suppose that a given involution is so defined. The product of an involution A and a projectivity will be called an *element* A. In case the F-points of A are drawn from a given set this will be indicated by giving A corresponding subscripts. We shall define a *regular Cremona transformation in S_k* to be one which can be expressed as a product of elements A. If for two such transformations the products can be formed in such a way that the number of elements A in each is the same and the arrangement of the F-points in forming

the product is the same then we shall say that the two are of the same *type*. Since the inverse of a regular transformation is likewise regular and the product of two regular transformations is regular we see that

(13) *The regular Cremona transformations in S_k form a group, the regular Cremona group in S_k.*

We shall understand by an F-point of a regular transformation, as distinguished from a point on an F-space, a point such that the correspondents of points consecutive to it lie on an f-space of dimension $k-1$. Though the regular transformations may have F-spaces of dimension greater than zero, as do the involutions A themselves, yet they have the property in common with C_m^2's that they are determined by their two sets of ρ F-points and by $k+2-\rho$ ordinary pairs if $\rho < k+2$. This statement is true for a projectivity and for an element A. If we assume that it is true for a regular transformation C which can be expressed as a product of $\mu - 1$ elements A then it is easy to see that it is true of the product of C and an element A and therefore holds generally.

That, as assumed above, the number of F-points of the regular transformation C_m^k is the same as that of its inverse $(C_m^k)^{-1}$ can be proved by a similar induction. Let $C_m^k = CA$ where C which sends x into x' has the set $p_1, \cdots, p_\sigma$ of F-points and C^{-1} has the set $q_1, \cdots, q_\sigma$ of F-points. Let A which sends x' into x'' have F-points, $r_1, \cdots, r_{k+1}$ of which τ are found in the set q and the remaining $k+1-\tau$ under C^{-1} correspond to ordinary points of C; and let $s_1, \cdots, s_{k+1}$ be the F-points of A^{-1}. Then the number of F-points of C_m^k is $\rho = (\sigma) + (k+1-\tau) - \mathrm{X}$ where X is the number of F-points of C which correspond to f-spaces of A^{-1} or it is the number of F-points of C which form with F-points of A^{-1} ordinary pairs of the product. The F-points of $(C_m^k)^{-1}$ are those of $A^{-1}C^{-1}$. For this product the numbers σ and $k+1$ interchange while τ and X remain the same whence ρ is the same.

A general set of points P_n^k will be said to be *congruent* to a set Q_n^k under the regular Cremona transformation C_m^k in S_k with $\rho \leqq n$ F-points if the F-points of C_m^k and of $(C_m^k)^{-1}$ are found in P_n^k and Q_n^k respectively and if the remaining $n-\rho$ of the points in each set form $n-\rho$ corresponding pairs of C_m^k. The *type* of congruence is determined by the type of C_m^k. If Q_n^k is projective to $Q_n'^k$ then P_n^k is congruent to $Q_n'^k$ under the transformation $C_m^k \pi$ of the same type as C_m^k. Thus congruence is both a mutual and a projective property of the sets.

If the set P_n^k is congruent to Q_n^k under C_m^k and if Q_n^k is congruent to R_n^k under D_μ^k then P_n^k is congruent to R_n^k under a properly arranged product $C_m^k \cdot D_\mu^k$. The argument used above for the product CA if account be taken of ordinary corresponding pairs under both C and A proves this statement for the case when D_μ^k is an element A. By expressing D_μ^k as a product of elements

A and by using the less general statement as a lemma the original statement can be proved.

If the type of C_m^k be given it can be expressed as a product of elements A. If two sets are congruent under an element A the coördinates of the one can be expressed rationally in terms of the coördinates of the other. Moreover this expression is unique if $k+1$ points of each set be placed at a given base. Evidently the same property holds for a product of elements A so that if P_n^k is congruent in some order to Q_n^k the coördinates of Q_n^k are rational in those of P_n^k.

If we map the sets P_n^k upon the points P of a space $\Sigma_{k(n-k-2)}$ as suggested in I, § 6 (53), then most of the conclusions above are embraced in the theorem:

(14) *If the sets P_n^k and Q_n^k are congruent in some order under regular transformation in S_k their maps P and Q in $\Sigma_{k(n-k-2)}$ are corresponding points under a Cremona transformation in Σ. Two sets congruent to a third are congruent to each other. The aggregate of projectively distinct sets Q_n^k congruent in some order to a given set P_n^k is mapped in Σ by an aggregate of points Q which form a conjugate set under the extended Cremona group $G_{n,k}$ in Σ.*

By using considerations entirely similar to those of § 3 the theorems (6) and (9) of § 3 can be generalized as follows:

(15) *The extended group $G_{n,k}$ in $\Sigma_{k(n-k-2)}$ can be generated by the $G_{n!}$ of P_n^k (whose generators are given in* I, § 7 (64)) *and the involutory transformation $\bar{A}$ obtained by inverting the variables in Σ. The $G_{n,k}$ contains subgroups isomorphic with $G_{n',k}$ if $n' < n$.*

This isomorphism is simple except in the cases where congruence of sets $P_{n'}^k$ implies projectivity in some order. In case $n' = k+3$ the isomorphism is 1 to 2^{k+2}.

The generator $\bar{A}$ determined by the element A is conjugate under $G_{n!}$ to the involution of type $\bar{A}$ determined by any $k+1$ of the points of P_n^k and it may therefore be replaced by any such involution. From the form of the associated sets P_n^k, Q_n^{n-k-2} given in I, § 6, p. 182, it is clear that the groups $G_{n!}$ coincide for the two sets and the generator $\bar{A}_{1,2,\cdots,k+1}$ of the one coincides with the generator $\bar{A}_{k+2,\cdots,n}$ of the other. Hence

(16) *The extended groups $G_{n,k}$ and $G_{n,n-k-2}$ of the associated point sets P_n^k and Q_n^{n-k-2} coincide.*

The various point sets can be arranged so that those whose norm

$$N = 2k + 2 - n$$

is zero lie in the principal diagonal of the array and those whose norms differ in sign are symmetrical with regard to this diagonal. An examination of this array with (15) and (16) in mind leads to the result:

(17) *The group $G_{n,k}$ contains subgroups isomorphic with $G_{n',k'}$, if either $k' \leqq k$ and $n'-k' < n-k$ or $k' < n-k-2$ and $n'-k' \leqq n-k$.*

We prove in § 6 that the number of types of congruence for P_9^2 and P_8^3 is infinite. Hence the number of types of congruence for sets of the array

$$\begin{array}{cccc} P_6^2 & P_7^2 & P_8^2 & \cdots \\ P_7^3 & P_8^3 & \cdots & \cdots \\ P_8^4 & \cdots & \cdots & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{array}$$

beyond P_8^2 in the first row and beyond P_7^3 in the second row is infinite. According to (16) this is true also of sets in the first column below P_8^4 and sets in the second column below P_7^2. Then according to (15) it is true of all the sets in all the rows beyond P_8^2, P_7^3, and P_8^4.

The proof in § 3 that the sets P_n^2 ($n \geqq 9$) congruent under C_m^2 are projectively distinct was based first on the fact that if P_n^2 was on an elliptic norm-curve the congruent set was on a birationally equivalent elliptic norm-curve, and second on the fact that C_m^2 had F-points of different types. Since the element A transforms elliptic $E^{(k+1)}$'s on P_n^k into elliptic $E^{(k+1)}$'s on Q_n^k, the transformation C_m^k with F-points at P_n^k has the same property. Moreover we shall see in § 5 (12) that no C_m^k has symmetrical F-points except those determined by the following sets for $n > k+3$: P_6^2; P_7^2, P_7^3; P_8^2, P_8^4. Thus in all further cases congruence of general sets implies non-projectivity.

(18) *The only finite groups $G_{n,k}$ ($n > k+3$) are the $G_{6,2}$, the $G_{7,2} = G_{7,3}$, and the $G_{8,2} = G_{8,4}$ which are identified in* § 3 (9). *All other groups $G_{n,k}$ are infinite and discontinuous.*

In the next two paragraphs groups isomorphic with $G_{n,k}$ are studied and the results are used to develop further properties of the regular transformations C_m^k and of the groups $G_{n,k}$.

5. The collineation group $g_{n,k}$

There is determined by the regular transformations associated with the set P_n^k a collineation group $g_{n,k}$ isomorphic with the Cremona group $G_{n,k}$ in $\Sigma_{n(n-k-2)}$. An element of $g_{n,k}$ describes the effect on spreads in S_k of a regular transformation C whose F-points are found in P_n^k. If a spread of order x_0 in S_k has an x_i-fold point at p_i it is transformed by C into a spread of order x_0' which has an x_j'-fold point at q_j ($i, j = 1, \cdots, n$). Given the type of C and the order of the set P_n^k with reference to Q_n^k this relation between x and x' must be unique, linear, and reversible, whence it is a proper collineation. Since C can be generated by a sequence of elements A, the $g_{n,k}$ is generated by

$$\begin{aligned} g_{n!}\colon \quad & x_0' = x_0, \quad x_i' = x_j; && (i, j = 1, \cdots, n); \\ A\colon \quad & x_0' = && kx_0 - x_1 - x_2 - \cdots - x_{k+1}, \\ & x_1' = (k-1)x_0 && - x_2 - \cdots - x_{k+1}, \end{aligned}$$

$$
\begin{aligned}
x_2' &= (k-1)x_0 - x_1 \quad - \cdots - x_{k+1},\\
&\cdots\cdots\cdots

C_{13}

		3	4
	13	-4	-3
3	8	$-3, -2$	-2
4	6	-2	$-2, -1$

C_{11}

		1	4	2
	11	-4	-3	-2
1	8	-3	-2	-2
4	6	-2	$-1, -2$	-1
2	4	-2	-1	$-1, 0$

C_9

		3	3	1
	9	-3	-2	-1
3	6	-2	$-2, -1$	-1
3	4	$-2, -1$	-1	0
1	2	-1	0	0

C_9'

		1	6
	9	-4	-2
1	8	-3	-2
6	4	-2	$0, -1$

C_{15}:

		7
	15	-4
7	8	$-3, -2$

Here the numbers outside of the diagram indicate the number of rows and columns of the matrix. The corresponding rectangles of the matrix are to be filled with the numbers given within. If two numbers are given the principal diagonal of the corresponding square is to be filled with the first of the two and the other places with the last of the two numbers.

The type C_{15} is a symmetric type listed in (30). Congruence under this type implies projectivity. It pairs off the remaining types by multiplication the pairs being easily located above. It is itself paired with the collineation type of C. The number of elements of the various types obtained by permuting the rows (or columns) is $2\binom{7}{4}$, $2\binom{7}{1}\binom{6}{2}$, $2\binom{7}{1}\binom{6}{3}$, $2\binom{7}{1}$, $2\binom{7}{0}$, i. e., 576 in all whence the order of $g_{7,3}$ is $2.288.7!$. Allowing for the projectivity under C_{15} the order of $G_{7,3} = G_{7,2}$ is only $288.7!$. Thus $g_{7,3}$ and $G_{7,3}$ are in 2 to 1 isomorphism.

Any invariant form of $g_{n,k}$ is symmetric in $x_1, \cdots, x_n$. It is easy to verify that the linear form

$$L \equiv (k+1)x_0 - (x_1 + \cdots + x_n)$$

is unaltered by A and therefore is an absolute invariant of $g_{n,k}$. An invariant quadratic form can be combined with L^2 so that no term in x_0^2 appears.

Assuming that it is $\alpha x_0 \sum x_1 + \beta \sum x_1^2 + \gamma \sum x_1 x_2$ and applying A we find that $\alpha = -2(k^2-1)$, $\beta = k(k+3)$, $\gamma = 2(k-1)$. By adding a proper multiple of L^2 the terms in $x_0 \sum x_1$ and $\sum x_1 x_2$ are eliminated and the invariant quadratic form is

$$M \equiv (k-1)x_0 - (x_1^2 + \cdots + x_n^2).$$

The $g_{n,k}$ has an invariant point also. For an S_{k-1} becomes under the involution A in S_k a $k$ic spread with $(k-1)$-fold points so that a $(k+1)$ic spread with $(k-1)$-fold points at P_n^k is transformed into a spread of the same sort. Hence the point

$$O \equiv k+1, \quad k-1, \quad \cdots, \quad k-1,$$

is invariant under $g_{n,k}$.

(21) *The $g_{n,k}$ has for absolute invariant the quadratic form M, as well as the point O and linear form L which are pole and polar as to M.*

There are three particular cases of interest here, namely those for which O and L are incident. This occurs when $(k+1)^2 - n(k-1) = 0$ or $n = k+3+4/(k-1)$. Since n must be an integer, $k-1 = 1, 2, 4$; $n = 9, 8, 9$. These cases P_9^2, P_8^3, P_9^5 have the further peculiarity that they are the only point sets which lie on a unique elliptic norm curve. For the $E^{(k+1)}$ in S_k has $(k+1)^2$ constants and it is $(k-1)$ conditions that it be on a point whence n must be $(k+1)^2/(k-1)$ if the number of $E^{(k+1)}$'s on P_n^k is to be finite.

The generator A and the transpositions of $g_{n,k}$ are such that for them $m-1$ and the s_j contain $k-1$ as a factor. If this is true of two elements of $g_{n,k}$ with coefficients m, m'; s_j, s_j'; r_i, r_i'; $\alpha_{ji}, \alpha_{ji}'$ it is true of the product. For $m'' = mn' - \sum r_i' s_i$ and

$$m'' - 1 = (m-1)(m'-1) + (m-1) + (m'-1) - \sum r_i' s_i;$$

and $s_j'' = s_j' m - \sum_{i=1}^{i=n} \alpha_{ji}' s_i$. Hence this factor will always appear in the m and s_j of (20) and it will be convenient to set

$$m - 1 = (k-1)\mu, \qquad s_j = (k-1)\sigma_j, \qquad r_i = \rho_i.$$

Then the general element of $g_{n,k}$ has the matrix

$$(22) \qquad \begin{pmatrix} (k-1)\mu + 1 & -\rho_1 & \cdots & -\rho_n \\ (k-1)\sigma_1 & -\alpha_{11} & \cdots & -\alpha_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ (k-1)\sigma_n & -\alpha_{1n} & \cdots & -\alpha_{nn} \end{pmatrix}.$$

This matrix takes the following more symmetric form under the substitution $\bar{x}_0 = i\sqrt{k-1}\, x_0$:

$$(23)\qquad \begin{pmatrix} (k-1)\mu+1 & -i\sqrt{k-1}\,\rho_1 & \cdots & -i\sqrt{k-1}\,\rho_n \\ -i\sqrt{k-1}\,\sigma_1 & -\alpha_{11} & \cdots & -\alpha_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ -i\sqrt{k-1}\,\sigma_n & -\alpha_{n1} & \cdots & -\alpha_{nn} \end{pmatrix}.$$

The invariant quadratic form M now is

$$\overline{M} = \bar{x}_0^2 + x_1^2 + \cdots + x_n^2,$$

whence the matrix (23) is orthogonal. Hence the determinant of (23) is $\epsilon = \pm 1$ and the minor of any element is ϵ times the element. The inverse matrix is obtained by interchanging rows and columns. On transforming back to x_0 we find that the matrix inverse to (22) is obtained by interchanging the ρ's with the σ's and the α_{ji} with the α_{ij}. But this inverse matrix itself belongs to an element of $g_{n,\,k}$ which corresponds to the inverse Cremona transformation in S_k. This shows that

(24) *For a regular Cremona transformation in S_k the direct and inverse transformation have the same order and the same number of F-points, and the numbers ρ_i, α_{ji}, σ_j for the one are the numbers σ_j, α_{ij}, ρ_i for the other.*

If we make use of the fact that L, M, and O are absolutely invariant under the element of $g_{n,\,k}$ whose matrix is (22) we get certain relations on the integer coefficients. If in these relations we transpose the coefficients according to (24) new relations are obtained. The entire set of relations is as follows:

$$(25)\qquad \begin{aligned} &\textstyle\sum_j \sigma_j = (k+1)\mu, && \textstyle\sum_j \sigma_j^2 = (m+1)\mu, \\ &\textstyle\sum_j \alpha_{ji} = (k+1)\rho_i - 1, && \textstyle\sum_j \alpha_{ji}^2 = (k-1)\rho_i^2 + 1, \\ &\textstyle\sum_j \alpha_{ji}\sigma_j = \rho_i m, && \textstyle\sum_j \alpha_{ji}\alpha_{jk} = (k-1)\rho_i\rho_k; \\ &\textstyle\sum_i \rho_i = (k+1)\mu, && \textstyle\sum_i \rho_i^2 = (m+1)\mu; \\ &\textstyle\sum_i \alpha_{ji} = (k+1)\sigma_j - 1, && \textstyle\sum_i \alpha_{ji}^2 = (k-1)\sigma_j^2 + 1, \\ &\textstyle\sum_i \alpha_{ji}\rho_i = \sigma_j m, && \textstyle\sum_i \alpha_{ji}\alpha_{ki} = (k-1)\sigma_j\sigma_k. \end{aligned}$$

These relations reduce in S_2 to the well-known set given in Clebsch-Lindemann (loc. cit.). The further facts there proven with regard to the sets of equal numbers ρ_i and σ_j and to the rectangles and squares in the matrix $\| \alpha_{ji} \|$ determined by these sets could be extended readily to the regular transformations in S_k.

It was noted in § 4 that $G_{n,\,k}$ and $G_{n,\,n-k-2}$ were identical since for proper arrangement of the points of P_n^k and Q_n^{n-k-2} the transpositions (ij) of $G_{n!}$ were the same in Σ and the generators $A_1, \ldots, A_{k+1}$ and $A_{k+2}, \ldots, A_n$ respectively also were the same in Σ. Since $g_{n,\,k}$ and $g_{n,\,n-k-2}$ are isomorphic with these groups they are in natural isomorphism with each other. We

might expect therefore to find that the one is merely a transform of the other. In order to prove this consider the involutions which generate $g_{n,k}$. They all are of the point $-S_{n-1}$ type. The invariant point of a transposition (ij) is $x_i = -x_j = 1$, $x_l = 0$; the S_{n-1} is $x_i - x_j = 0$. The invariant point of A is $x_0 = x_1 = \cdots = x_{k+1} = 1$, $x_{k+2} = \cdots = x_n = 0$; the S_{n-1} is

$$(k-1)x_0 - x_1 - \cdots - x_{k+1} = 0.$$

Hence the point of each is on L and the S_{n-1} of each is on O or

(26) *Every element of $g_{n,k}$ is interchangeable with every element of the one-parameter group of homologies determined by O, L.* Thus if $g_{n,k}$ and $g_{n,n-k-2}$ are conjugate they are conjugate in infinitely many ways. A collineation T which sends the transpositions of $g_{n,k}$ into the corresponding ones of $g_{n,n-k-2}$ must have the form

$$T: \quad x_0 = \alpha\bar{x}_0 + \beta\bar{\sigma}, \qquad \bar{\sigma} = \textstyle\sum_i \bar{x}_i$$

$$x_i = \gamma\bar{x}_0 + \epsilon\bar{x}_i + \delta\bar{\sigma}, \qquad i = (1, \cdots, n).$$

If T sends $(k-1)x_0 - x_1 \cdots - x_{k+n}$ into $(n-k-3)\bar{x}_0 - \bar{x}_{k+2} - \cdots - \bar{x}_n$ then $(k-1)\alpha - (k+1)\gamma + \epsilon(n-k-3) = 0$ and

$$\epsilon = (k-1)\beta - (k+1)\delta.$$

If it sends the fixed point of $A_1, \ldots, {}_{k+1}$ into that of $A_{k+2}, \ldots, {}_n$ then

$$\alpha + (n-k-1)\beta + \epsilon = 0, \qquad \gamma + (n-k-1)\delta + \epsilon = 0.$$

In order to get a definite form for T let $\delta = 0$ and $\epsilon = k-1$. Then $\gamma = -(k-1)$, $\beta = 1$, and $\alpha = -(n-2)$. These coefficients are valid for all values of n and k with which we are concerned though the theorem (26) admits the three exceptions mentioned under (21). To within a factor of proportionality we get

$$T^{-1}: \qquad \bar{x}_0 = (k-1)x_0 - \sigma,$$

$$\bar{x}_i - \bar{x}_0 = 2x_i.$$

The determinant of T is $2(k-1)^n$ and of T^{-1} is $2^n(k-1)$. If then we form the product T (22) T^{-1}, where (22) is the matrix above, and divide each coefficient by $2(k-1)$ the resulting element is that one of $g_{n,n-k-2}$ which corresponds in the above isomorphism to the element (22) of $g_{n,k}$.

(27) *The group $g_{n,k}$ is transformed into its associated $g_{n,n-k-2}$ by the substitution T^{-1}. The element of $g_{n,n-k-2}$ which corresponds to the element* (22) *of $g_{n,k}$ has the matrix*

$$\begin{pmatrix} (n-k-3)\mu + 1 & -(\mu - \rho_i) \\ (n-k-3)(\mu - \sigma_j) & -(\mu - \rho_i - \sigma_j + \alpha_{ji}) \end{pmatrix}.$$

These corresponding elements of $g_{n,k}$ *and* $g_{n,n-k-2}$ *lead to the same element in* $G_{n,k} = G_{n,n-k-2}$.

Since the α_{ij} are usually positive integers or zero but may for ordinary points have the value -1 we see that

(28) *For every regular Cremona transformation in* S_k *the inequalities*

$$\sigma_j + \rho_i \leqq \mu + 1 + \alpha_{ji}$$

are valid. In particular for S_2 $\sigma_j + \rho_i \leqq m + \alpha_{ji}$.

This inequality for S_2 is known but the above proof based on point sets in higher dimensions seems rather interesting.

We shall now determine the types of regular Cremona transformations in S_k with a single symmetrical set of F-points. For the corresponding element of $g_{n,k}$, $\rho_i = \sigma_j = \rho$, $\alpha_{ii} = \alpha$, and $\alpha_{ij} = \beta \neq \alpha$. The relations (25) now become

$$n\rho = (k+1)\mu, \qquad n\rho^2 = \mu[(k-1)\mu + 2],$$

$$(n-1)\beta + \alpha = (k+1)\rho - 1, \qquad (n-1)\beta^2 + \alpha^2 = (k-1)\rho^2 + 1,$$

$$(n-1)\beta + \alpha = (k-1)\mu + 1, \qquad (n-2)\beta^2 + 2\alpha\beta = (k-1)\rho^2.$$

From the last two $\alpha = \beta \pm 1$ and the set can be replaced by

$$n\beta^2 \pm 2\beta = (k-1)\rho^2, \qquad (k \pm 1)\rho = (k-1)\mu + 2,$$

$$n\beta \pm 1 = (k-1)\mu + 1, \qquad n\rho = (k+1)\mu$$

Eliminating ρ from the second and fourth of these we have

$$[(k+1)^2 - n(k-1)]\mu = 2n.$$

Since n, k, μ are positive integers we have

$$1 \leqq [(k+1)^2 - (k-1)n] \leqq 2n$$

whence $k+1 \leqq n \leqq k+3+3/(k-1)$, or $k+1 \leqq n < k+6$. Taking for n values from $k+6$ to $k+1$ in order we find that if

(a) $n = k+6$, $\mu = (2k+12)/(-3k+7)$ whence since $\mu > 0$ and $k \leqq 2$, $k = 2$, $\mu = 16$, $n = 8$, $\rho = 6$, $\beta = 2$, $\alpha = 3 : P_8^2$;

(b) $n = k+5$, $\mu = (2k+10)/(-2k+6)$ whence $k = 2$, $\mu = 7$, $n = 7$, $\rho = 3$, $\beta = 1$, $\alpha = 2 : P_7^2$;

(c) $n = k+4$, $\mu = (2k+8)/(-k+5)$ whence

(c_1) $k = 4$, $\mu = 16$, $n = 8$, $\rho = 10$, $\beta = 6$, $\alpha = 7 : P_8^4$;

(c_2) $k = 3$, $\mu = 7$, $n = 7$, $\rho = 4$, $\beta = 2$, $\alpha = 3 : P_7^3$;

(c_3) $k=2, \mu=4, n=6, \rho=2, \beta=1, \alpha=0 : P_6^2$;

(d) $n=k+3,\ \mu=(k+3)/2,\ \rho=(k+1)/2,\ \beta=(k-1)/2,$
$\alpha=(k+1)/2 : P_{k+3}^k$ k odd;

(e) $n=k+2,\ \mu=2(k+2)/(k+3)=2-2/(k+3)$ which is impossible;

(f) $n=k+1, \mu=1, \rho=1, \beta=1, \alpha=0 : P_{k+1}^k$.

So far as the existence of the corresponding Cremona transformations is concerned we see that the type (f) is the element A, and the types (a), (b), (c_3) were discussed in § 2. The type (d) for $k=3$ is the C_7' listed at the beginning of this paragraph. All these are well known but the types (c_1),* (c_2), and (d) for general k seem new to the literature. The existence of the types (c_1) and (c_2) is a consequence of the existence of their associated types (a) and (b). To show that (d) exists for every odd value of k let $k=2\nu+1$, let P_{k+3}^k be $p_1, \cdots, p_{2\nu+4}$, and let I_{ij} be the involution A which interchanges p_i, p_j and which has the other $2\nu+2$ points as F-points. Then it is not hard to verify that the transformation $I_{12}\, I_{34} \cdots I_{2\nu+3,\, 2\nu+4}$ is of the required type (d). If the properties of the involutory type (d) as known for $k=3$ be generalized we get a new generalization of the Kummer and Weddle surfaces.

(29) *The only point sets which serve in a symmetrical way as F-points to define a regular Cremona transformation are* P_{k+1}^k ($k \geqq 2$); P_{k+3}^k (k *odd*); P_6^2; P_7^2, P_7^3; *and* P_8^2, P_8^4.

The elements of $g_{n,\,k}$ corresponding to these symmetric types are necessarily involutory. In most of the cases this involution is the harmonic perspectivity I determined by O and L whose matrix is

$$\text{I:}\quad \begin{pmatrix} n(k-1)+(k+1)^2 & -2(k+1) & -2(k+1) \\ 2(k-1)^2 & -[(k+1)^2-(n-2)(k-1)] & -2(k-1) \\ 2(k-1)^2 & -2(k-1) & -[(k+1)^2-(n-2)(k-1)] \end{pmatrix}.$$

If this is to leave L absolutely unaltered all the coefficients must be divided by $[(k+1)^2 - n(k-1)]$. For P_8^2, P_8^4 and P_7^2, P_7^3 this factor is 1 and I belongs to $g_{n,\,k}$. For P_{k+3}^k the factor is 4 and again I belongs to $g_{n,\,k}$. For P_{k+1}^k and P_6^2 however the symmetrical element is found in the conjugate set of generators and I does not belong to $g_{n,\,k}$.

(30) *The element of* $g_{n,\,k}$ *corresponding to the above symmetrical types is the involution I determined by O and L when* $\alpha > \beta$; *when* $\alpha < \beta$ *it is found in the conjugate set of generators.*

The transpositions of $g_{n!}$ and the element A in (19) all have the determinant -1 whence

* For the type (c_1) cf. Conner, loc. cit.

(31) *The group $g_{n,k}$ has an invariant subgroup of index two which consists of all of its elements of determinant unity.*

6. The group $e_{n,k}$

The group $g_{n,k}$ of § 5 represented the effect of regular Cremona transformations in S_k upon spreads of dimension $k-1$. The group $e_{n,k}$ of this paragraph represents the transition from a set P_n^k given by its elliptic parameters on an elliptic norm curve E^{k+1} in S_k to a congruent set Q_n^k similarly given on the same curve. This requires in general that P_n^k be a special set but the resulting transformations on the parameters are linear and their group $e_{n,k}$ is isomorphic with $G_{n,k}$. We shall find that in certain cases this new group is very useful.

The element A in S_k transforms a curve of order g with h_i-fold points at p_i $(i=1, \cdots, k+1)$ into a curve of order $(k+1)g - k\sum_{i=k}^{i=k+1} h_i$ with $[g + h_i - \sum_{j=1}^{j=k+1} h_j]$-fold points at q_i. It therefore transforms an elliptic curve E^{k+1} with simple points at P_n^k into an elliptic curve E'^{k+1} with simple points at Q_n^k. Since E and E' have the same modulus and are normal curves in S_k each invariant under a collineation group of order $2(k+1)^2$ there will be a collineation K—any one of $2(k+1)^2$ possible collineations—which will transform E' back into E. Let v be the elliptic parameter on E such that $v_1 + \cdots + v_{k+1} \equiv 0 \pmod{\omega}$ is the condition that $k+1$ points of E lie on an S_{k-1}. Let $v = u_1, \cdots, u_n$ be the parameters on E of P_n^k whose first $k+1$ points are the F-points of A. Let the point v of E be transformed by A into the point v of E' and let the point v of E' be transformed by K into the point v' of E. Then the $k+1$ points $v_1', \cdots, v_{k+1}'$ lie on an S_{k-1} if the $k+1$ points $v_1, \cdots, v_{k+1}$ lie on a $k$ic spread with $(k-1)$-fold points at $u_1, \cdots, u_{k+1}$. Hence $v_1' + \cdots + v_{k+1}' \equiv v_1 + \cdots + v_{k+1} + (k-1)(u_1 + \cdots + u_{k+1})$. Thus $v' = v + (u_1 + \cdots + u_{k+1})(k-1)/(k+1)$ represents the effect of AK upon an ordinary point of A. But for an F-point of A, say $v = u_i$ on E, the corresponding point on E' is not $v = u_i$ but rather the corresponding point q_i of Q_n^k whose parameter is $v_i = u_i - (u_1 + \cdots + u_{k+1})$. For this point therefore $v_i' = u_i - (u_1 + \cdots + u_{k+1})\,2/(k+1)$, $(i = 1, \cdots, k+1)$. Since Q_n^k is congruent to P_n^k under A the transform of Q_n^k by the collineation K is also congruent to P_n^k whence the set on E^{k+1} determined by parameters u', where

$$
\begin{aligned}
A:\quad & u_i' \equiv u_i - (u_1 + \cdots + u_{k+1})\,2/(k+1) && (i = 1, \cdots, k+1), \\
& u_j' \equiv u_j + (u_1 + \cdots + u_{k+1})(k-1)/(k+1) && (j = k+2, \cdots, n),
\end{aligned} \tag{32}
$$

is congruent to the set u on E^{k+1} under the regular transformation of type A.

It should be noted that a set u_i on E is projectively equivalent to the

$2(k+1)^2$ sets, $\pm u_i + \omega/(k+1)$, where ω is any period. If the sign of all the u's be changed, the signs of the u''s in (32) are changed. If the u's be increased by $\omega/(k+1)$, $(u_1 + \cdots + u_{k+1})$ is unaltered and the u''s each are increased by $\omega/(k+1)$. In any case $(u_1 + \cdots + u_{k+1})/(k+1)$ is indeterminate to within $\omega/(k+1)$. Thus (32) might be regarded as a relation between the two sets of $2(k+1)^2$ projectively equivalent P_n^k's, $\pm u_i + \omega/(k+1)$ and $\pm u_i' + \omega/(k+1)$. But either set can be determined by a sample and it is simpler to drop the congruence sign and to use the equality sign. Then we have merely a linear transformation from the set u to the congruent set u'.

The process by which we combine elements A to obtain sets congruent to P_n^k is represented on E^{k+1} by the process of combining linear transformations of type A and a change of the order of the points is represented by a change of the order of the parameters whence

(33) *The totality of sets Q_n^k congruent in some order to a given set P_n^k on E^{k+1} with paramaters $u_1, \cdots, u_n$ is obtained on E^{k+1} by effecting on the parameters u the operations of the group $e_{n,k}$ generated by transpositions of the parameters and by A in* (32). *These generators form part of a conjugate set of involutory elements. The group $e_{n,k}$ of linear transformations of determinant ± 1 is isomorphic with $G_{n,k}$ and is simply isomorphic with $g_{n,k}$.*

This simple isomorphism with $g_{n,k}$ is due to the fact that each describes the result of regular transformations C_m^k in S_k.

The transpositions are point-S_{n-1} involutions. The generator A is of the same sort. Every point of $u_1 + \cdots + u_k = 0$ is fixed under A and the point $u_i = 1, u_j = -(k-1)/2\,(i = 1, \cdots, k+1; j = k+2, \cdots, n)$ is changed in sign under A. The group $e_{n,n-k-2}$ associated with Q_n^{n-k-2} is as before in natural isomorphism with $e_{n,k}$ and we should expect that the one is the transform of the other in such a way that $e_{n!}$ is the same for both and $A_{1,\ldots,k+1}$ of $e_{n,k}$ is transformed into $A_{k+2,\ldots,n}$ of $e_{n,n-k-2}$. The transformation

$$T: \qquad u_i = -\bar{u}_i + \bar{\sigma}/(k+1) \qquad (\bar{\sigma} = \bar{u}_1 + \cdots + \bar{u}_n)$$

will send $u_1 + \cdots + u_{k+1}$ into $\bar{u}_{k+2} + \cdots + \bar{u}_n$; and

$$T^{-1}: \qquad \bar{u}_i = -u_i + \sigma/(n-k-1)$$

sends the fixed point of $A_{1,\ldots,k+1}$ into that of $A_{k+2,\ldots,n}$. Hence

(34) *The group $e_{n,k}$ of P_n^k is transformed by the substitution, $u_i = \bar{u}_i + \bar{\sigma}/(k+1)$ into the group $e_{n,n-k-2}$ of Q_n^{n-k-2}. Corresponding elements of the two groups belong to the same element of $G_{n,k} = G_{n,n-k-2}$.*

An invariant form of $e_{n,k}$ must be symmetric in $u_1, \cdots, u_n$. If it admits also the element A it is unaltered by the entire group. If then we test the linear and quadratic forms we find that

(35) *The group $e_{n,k}$ has the absolute quadratic invariant*

$$[(k+1)^2 - n(k-1)](u_1^2 + \cdots + u_n^2) + (k-1)(u_1 + \cdots + u_n)^2.$$

For the cases P_9^2, P_9^5; P_8^3 only has $e_{n,k}$ the invariant linear form $\sigma = u_1 + \cdots + u_n$.

Of the two elements of $g_{n,k}$ and $e_{n,k}$ which are in natural correspondence the former is more explicit concerning the properties of C_m^k while usually, as we shall see, the latter has a more convenient form. It is then worth while to find one element in terms of the other. Let the element of $e_{n,k}$ determined by the element (20) of $g_{n,k}$ be

$$u_j' = \sum_{i=1}^{i=n} \beta_{ji} u_i \qquad (j = 1, \cdots, n). \tag{36}$$

This is determined first from the fact that $k+1$ points of E'^{k+1} on an S_{k-1} correspond to the meets of E^{k+1} with an m-ic spread. This leads to the relation

$$v_1' + \cdots + v_{k+1}' = v_1 + \cdots + v_{k+1} + (k-1)(\rho_1 u_1 + \cdots + \rho_n u_n)$$

or $v' = v + (\rho_1 u_1 + \cdots + \rho_n u_n)(k-1)/(k+1)$. But the parameter v_j on E^{k+1} which on E'^{k+1} determines the point q_j which corresponds to p_j is furnished by the extra meet with E^{k+1} of the f-spread determined by q_j, i. e., $v_j = -\sum_{i=1}^{i=n} \alpha_{ji} u_i$. Substituting this value for v above we find that

$$\beta_{ji} = (k-1)\rho_i/(k+1) - \alpha_{ji}. \tag{37}$$

This determines the element of $e_{n,k}$ in terms of the element of $g_{n,k}$. If on the other hand the β_{ji} are given we can first locate the zero values of ρ_i by noting the columns of β_{ji} whose elements all are zero except one which is unity. From the relations (25) § 5 we find that

$$\sum_j \beta_{ji} = \rho_i [n(k-1) - (k+1)^2]/(k+1) + 1.$$

Thus ρ_i is determined except when P_n^k is P_9^2, P_9^5, or P_8^3. In these cases we can use $\sum_j \beta_{ji}^2 = 2(k-1)(\rho_i^2 - \rho_i) + (3k-1)/(k+1)$. This equation in ρ_i is satisfied by a single positive integer ρ_i greater than zero. Knowing the values ρ_i we find from (37) the α_{ji} and again from (25) § 5 the m and σ_j.

The remainder of this paragraph is devoted to the particular cases of P_9^2, P_9^5, and P_8^3. Since P_9^2 and P_9^5 are associated sets we need to consider only the one. The sets P_9^2 and P_8^3 have an added interest in that they are the first sets for which $G_{n,k}$ is infinite. For these sets also $e_{n,k}$ has the absolute linear invariant σ.

Let us first investigate the conjugate set of irrational absolute invariants typified by $u_i - u_j$ whose vanishing implies that the two points p_i and p_j of P_9^2 on E^3 have become coincident. Of this kind there are 36 conjugate under $e_{9!}$. If on these we effect the transformation A we get a new kind

$u_i + u_j + u_k$ which vanishes if the three points are on a line. There are $\binom{9}{3} = 84$ of these conjugate under $e_{9,1}$. A further application of A leads to the type $u_1 + \cdots + u_6$ which vanishes if the six points are on a conic. If A be used again we get the type $2u_1 + u_2 + \cdots + u_8$ which vanishes if a rational cubic with node at u_1 passes through $u_2, \cdots, u_8$; etc., *ad infinitum*. We observe however that if $\sigma = 0$ the third type $u_1 + \cdots + u_6$ and the second type $u_7 + u_8 + u_9$ coalesce corresponding to the geometric fact that if P_9^2 is the base of a pencil of cubics and if three points are on a line then the remaining six points are on a conic. Similarly the fourth and the first type coalesce if $\sigma = 0$, a fact also geometrically evident. Let us say that the coalescent types are *congruent mod.* σ. We have therefore only 120 types incongruent mod. σ and these types are permuted by the elements of $e_{9,2}$. To identify this permutation group let us take the basis notation for the theta functions, $p = 4$. Let the E quadric Q_0 be isolated, let the type $u_i - u_j$ be associated with the O quadric Q_{0ij}, the type $u_i + u_j + u_k$ with the O quadric Q_{ijk}, the transposition (ik) with the involution I_{ik} determined by the points P_{ik}, and the element A_{ijk} with the involution I_{0ijk} $(i, j, k = 1, \cdots, 9)$. Then we find that the involutions I permute the 120 O quadrics just as the generators of $e_{9,2}$ permute the 120 types of invariants. Hence

(38) *The infinite system of irrational invariants conjugate to* $u_1 - u_2$ *under* $e_{9,2}$ *divide into* 120 *conjugate sets such that the infinite number in each set are congruent to each other mod.* σ. *Thus* $e_{9,2}$ *has an invariant subgroup* $i_{9,2}$ *of infinite order whose factor group* $f_{9,2}$ *has the order* $8!8640 = 9!960$ *and is isomorphic with the* $G_{8,2}$ *of* § 3 (12).

There remains to be considered the construction of the invariant subgroup $i_{9,2}$. Since the elements of $i_{9,2}$ leave $u_i - u_j$ invariant mod. σ to within sign they must be either involutory of the form $u_i' = -u_i + m_i\sigma$ or parabolic of the form $u_i' = u_i + l_i\sigma$ $(i = 1, \cdots, 9)$. Here $m_1 + \cdots + m_9 = 2$ and $l_1 + \cdots + l_9 = 0$ because σ is invariant. From (37) we see that the l and m are integers or fractions with denominator 3. Moreover since $u_i - u_j$ is altered by at most an integer multiple of σ, the differences of the l's or of the m's are integers. Hence

(39) *The group* $i_{9,2}$ *has an invariant abelian subgroup of index two whose elements are of the form* $u_i' = u_i + (\lambda_i - r/3)\sigma$; *the remaining elements of* $i_{9,2}$ *all are involutory of the form* $u_i' = -u_i + (\mu_i - r/3)\sigma$ $(i = 1, \cdots, 9)$. *Here* $r = 0, 1, 2$ *and* λ_i, μ_i *are integers such that* $\sum_i \lambda_i = 3r$ *and* $\sum_i \mu_i = 3r + 2$.

In order to prove that the arithmetical conditions of (39) are sufficient let us develop the subgroup of $e_{n,k}$ which arises from transformations C_m^2 with a symmetrical set of 8 F-points and one isolated F-point say p_9. If these have orders ρ_1 and ρ_2 then $8\rho_1^2 + \rho_2^2 = m^2 - 1$ and $8\rho_1 + \rho_2 = 3(m - 1)$. Eliminating ρ_2 we have $36\rho_1^2 = (m - 1)[24\rho_1 + 1 - 4(m - 1)]$ whence

$m - 1 = 4n$. Then $n = (3\rho_1 - 4n)^2$ or $n = l^2$ and $3\rho_1 = l(4l \pm 1)$. If we set $l = 3\nu$ or $l = 3\nu \mp 1$ we find two types of transformation, the ambiguous sign being accounted for by a change of sign of ν. These types are

$$D(\nu): \quad \begin{aligned} m &= 36\nu^2 + 1, \\ \rho_1 &= \nu(12\nu + 1), \\ \rho_2 &= 4\nu(3\nu - 2); \end{aligned} \qquad C(\nu): \quad \begin{aligned} m &= 4(3\nu + 1)^2 + 1, \\ \rho_1 &= (3\nu + 1)(4\nu + 1), \\ \rho_2 &= 4(3\nu + 1)(\nu + 1). \end{aligned}$$

To verify that these transformations exist we note that $C(-1)$ is E_{17} of § 2 which we denote here by E_9 to indicate that p_9 is an ordinary point; and that $C(0)$ denoted hereafter by F_9 is the known transformation of order 5 with a 4-fold point at p_9 and simple points at $p_1, \cdots, p_8$. By direct multiplication we find that $C(\nu)E_9 = D(\nu + 1)$ and $C(\nu)F_9 = D(\nu)$. Hence

$$D(\nu + 1)\,E_9 = C(\nu) \quad \text{and} \quad D(\nu)F_9 = C(\nu);$$

also $C(\nu)E_9F_9 = D(\nu + 1)F_9 = C(\nu + 1)$. Since $C(0) = F_9$ we have $C(\nu) = F_9(E_9F_9)^\nu = (F_9E_9)^\nu F_9$. Finally if we set $D(1) = D_9 = F_9E_9$ then $D(\nu) = (F_9E_9)^\nu = D_9^\nu$ and $C(\nu) = F_9D^{-\nu} = D_9^\nu F_9$. Hence

(40) *The types of C_m^2 with 8 symmetrical and one isolated F-point lead to elements of $e_{9,2}$ which lie in the invariant subgroup $i_{9,2}$ and constitute a dihedral subgroup of infinite order generated by the involutions F_9 and E_9 whose product D_9 is parabolic. In the dihedral group the generators belong to distinct conjugate sets.*

The last statement follows from the fact that D_9 transforms $C(\nu)$ into $C(\nu - 2)$. That all these elements lie in $i_{9,2}$ follows from the parametric expressions of F_9 and E_9 which from (37) are

$$F_9: \begin{aligned} u_1' &= -u_1 + \sigma/3, \\ &\cdots\cdots\cdots \\ u_8' &= -u_8 + \sigma/3, \\ u_9' &= -u_9 + \sigma/3 - \sigma; \end{aligned} \qquad E_9: \begin{aligned} u_1' &= -u_1, \\ &\cdots\cdots \\ u_8' &= -u_8, \\ u_9' &= -u_9 + 2\sigma; \end{aligned} \qquad D_9: \begin{aligned} u_1' &= u_1 - \sigma/3, \\ &\cdots\cdots\cdots \\ u_8' &= u_8 - \sigma/3, \\ u_9' &= u_9 - \sigma/3 + 3\sigma. \end{aligned}$$

We see that in D_9 the differences of the integers λ of (39) all are divisible by 3 and that this is true as well for any product $D_1^{\nu_1}D_2^{\nu_2}\cdots D_9^{\nu_9}$ where D_i is formed for the point p_i as D_9 for p_9. Hence we cannot hope to get in this way all possible elements of the form (39). However the product

$$C_{2,1} = E_2E_1: \begin{aligned} u_1' &= u_1 + 2\sigma, \\ u_2' &= u_2 - 2\sigma, \\ u_3' &= u_3, \\ &\cdots\cdots \\ u_9' &= u_9 \end{aligned}$$

is not subject to this defect. Then the product

$$D_2^{\nu_2} D_3^{\nu_3} \cdots D_9^{\nu_9} C_{2,1}^{\rho_2} C_{3,1}^{\rho_3} \cdots C_{2,9}^{\rho_9} \tag{41}$$

will be the general element of the abelian subgroup of (39) if

$$\begin{aligned}
-(\nu_2 + \cdots + \nu_9)/3 + 2(\rho_2 + \cdots + \rho_9) &= \lambda_1 - r/3, \\
3\nu_2 - (\nu_2 + \cdots + \nu_9)/3 - 2\rho_2 &= \lambda_2 - r/3, \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \\
3\nu_9 - (\nu_2 + \cdots + \nu_9)/3 - 2\rho_9 &= \lambda_9 - r/3.
\end{aligned}$$

These 9 equations are linearly related so that we need to satisfy only the last 8. Let $\lambda_j - \lambda_1 = 3a_j - 2b_j$ $(j = 2, \cdots, 9)$ where $b_j = 0, 1, 2$. Then $2\sum_j b_j = 3\sum_j a_j + 9\lambda_1 - 3r$ or $\sum_j b_j = 3s$, where s is an integer. If then we set $\rho_2, \cdots, \rho_9$ equal to $b_2, \cdots, b_9$ and $\nu_2 + \cdots + \nu_9$ equal to $a_2 + 2s$, $\cdots, a_9 + 2s$ respectively the equations are satisfied for all permissible integer values of λ and r. Hence

(42) *Every element of the form described in* (39) *actually occurs in* $i_{9,2}$. *Every element of the abelian subgroup can be expressed as a product* (41) *in one and only one way if we restrict the exponents* $\rho_2, \cdots, \rho_9$ *to be* $0, 1, 2$. *The remaining elements of* $i_{9,2}$ *can be expressed as a product of* E_1 *and* (41).

A consequence of this theorem is that the infinite number of types of ternary Cremona transformations with 9 F-points can be arranged in $960 \cdot 3^8 \cdot 2$ classes such that each class contains an infinite number of types depending on the unrestricted variation of the 8 integers $\nu_2, \cdots, \nu_9$.

We now turn to the set P_8^3 on E^4 with parameters $u_1, \cdots, u_8$. The group, $e_{8,3}$ is in most respects analogous to $e_{9,2}$ but we shall find some points of difference. The set of irrational invariants of $e_{8,3}$ conjugate to $u_1 - u_2$ is infinite in number. If however P_8^3 is self associated, i. e., if $\sigma = 0$, they reduce to a set of 63 which comprise 28 of the form $u_i - u_j$ and 35 of the form $u_i + u_j + u_k + u_l$. If in the finite geometry of the theta functions for $p = 3$ we identify these invariants with the 63 points P_{ij} and P_{ijkl} and if we identify the generators (ik) and A_{ijkl} of $e_{8,3}$ with the involutions on the 63 points, it is easy to verify that the invariants are permuted by the generators just as the points are permuted by the involutions. Hence

(43) *The infinite set of irrational invariants conjugate to* $u_1 - u_2$ *under* $e_{8,3}$ *divide into* 63 *conjugate sets such that the infinite number in each set are congruent mod.* σ. *The* $e_{8,3}$ *has an invariant subgroup* $i_{8,3}$ *of infinite order whose factor group* $f_{8,3}$ *is of order* 71288 *and simply isomorphic with* $G_{7,3} = G_{7,2}$.

To determine the construction of $i_{8,3}$ we note as before that its elements must be either parabolic of the form $u_i' = u_i + (\lambda_i - r/2)\sigma$ or involutory of the form $u_i' = -u_i + (\mu_i - r/2)\sigma$ $(i = 1, \cdots, 8)$, where λ_i and μ_i are

integers, r is 0, 1, and $\sum_i \lambda_i = 4r$, $\sum_i \mu_i = 4r+2$. In order to prove that these are the only conditions on the coefficients we use (25) § 5 to determine all types of regular Cremona transformations in S_3 with 8 F-points 7 of which form a symmetrical set. The algebra runs along about as in the preceding case. The resulting types, as elements of $g_{8,3}$ in the notation at the beginning of § 5 have the following matrices:

$C(\nu)$:

		1	7
1	$14(2\nu+1)^2+1$	$-7\nu(2\nu+1)$	$-(2\nu+1)(7\nu+4)$
7	$14\nu(2\nu+1)$	$-(7\nu^2-1)$	$-\nu(7\nu+4)$
	$2(2\nu+1)(7\nu+4)$	$-\nu(7\nu+4)$	$-(7\nu^2+8\nu+3)$, $-(7\nu^2+8\nu+2)$

;

$D(\nu)$:

		1	7
1	$56\nu^2+1$	$-7\nu(2\nu+1)$	$-\nu(14\nu-1)$
7	$14\nu(2\nu-1)$	$-(7\nu^2-1)$	$-\nu(7\nu-4)$
	$2\nu(14\nu-1)$	$-\nu(7\nu+4)$	$-(7\nu^2-1)$, $-7\nu^2$

.

That these types exist for all integer values of ν can be shown as follows. The type $C(0)$ is C_{15} of § 5 and $C^2(0) = 1$. The type $C(-1)$ also of order 15 and period two can be reduced to the known type C_5 of § 5 by multiplying it by a product of elements A. We find that $C(-1)C(0) = D(1)$ and that $D(\nu)D(1) = D(1)D(\nu) = D(\nu+1) = [D(1)]^{\nu+1}$. Thus $C(0)$ and $C(-1)$ generate a dihedral group in $g_{8,3}$ whose cyclic subgroup is generated by $D(1)$. If we denote $D(1)$ by D_1 and $C(0)$ by C_1 to indicate that the point p_1 is isolated we find from (37) that the corresponding elements of $e_{8,3}$ are

$$C_1: \quad u_1' = -u_1 + 2\sigma, \quad u_2' = -u_2, \quad \ldots, \quad u_8' = -u_8;$$

$$D_1: \quad u_1' = u_1 - \sigma/2 + 4\sigma, \quad u_2' = u_2 - \sigma/2, \quad \ldots, \quad u_8' = u_8 - \sigma/2;$$

$$C_{2,1} = C_2 C_1: \quad u_1' = u_1 + 2\sigma, \quad u_2' = u_2 - 2\sigma, \quad u_3' = u_3, \quad \ldots, \quad u_8' = u_8.$$

In all these elements the differences of the λ's and μ's are even so that some missing elements are yet to be found. The involutions A_{1234} and A_{5678} of

$e_{8,3}$ are the same mod. σ. Hence their product

$$u_i' = u_i + \sigma/2, \qquad u_j' = u_j + \sigma/2 - \sigma \qquad (i = 1, \cdots, 4;\ j = 5, \cdots, 8)$$

is found in $i_{8,3}$. It does not have the defect noted above. By combining a properly selected pair of such products we find the simple element of $i_{8,3}$:

$$E_{2,1}: \quad u_1' = u_1 + \sigma, \qquad u_2' = u_2 - \sigma, \qquad u_3' = u_3, \qquad \cdots, \qquad u_8' = u_8.$$

We can now prove as before the theorem:

(44) *The invariant subgroup $i_{8,3}$ of $e_{8,3}$ contains an invariant abelian subgroup of index two whose elements all are parabolic of the form $u_i' = u_i + (\lambda_i - r/2)\sigma$; the remaining elements of $i_{8,3}$ all are involutory of the form*

$$u_i' = -u_i + (\mu_i - r/2)\sigma$$

where λ_i, μ_i, r are any integers such that $\sum_i \lambda_i = 4r$, $\sum_i \mu_i = 4r + 2$, $r = 0, 1$ $(i = 1, \cdots, 8)$. Every element of the abelian subgroup can be expressed in a single way by a product

$$D_2^{\nu_2} D_3^{\nu_3} \cdots D_8^{\nu_8} C_{3,1}^{\nu_{31}} C_{4,1}^{\nu_{41}} \cdots C_{8,1}^{\nu_{81}} E_{3,1}^{\rho_{31}} E_{4,1}^{\rho_{41}} \cdots E_{8,1}^{\rho_{81}}$$

such that $\nu_{j,1}, \rho_{j,1} = 0, 1$ $(j = 3, \cdots, 8)$.

In fact if $\lambda_j - \lambda_2 = 4a_{j,2} - 2b_{j,2} - c_{j,2}$ where $b_{j,2}, c_{j,2} = 0, 1$ then the exponents in the product are precisely $\rho_{j1} = c_{j,2}$, $\nu_{j,1} = b_{j,2}$, $\nu_2 = k$, $\nu_j = k + a_{j,2}$ where $k = \sum_j a_{j,2} + 2\lambda_2 - r$.

By means of the foregoing theorems properties of the groups $G_{8,3}$ and $G_{9,2}$ become accessible and further theorems in regard to particular point sets can be established. Thus we see at once from (43) and (44) that

(45) *If P_8^3 is the set of base points of a net of quadrics there are only* 36 *projectively distinct sets congruent in some order to P_8^3.*

This could be inferred from the theory of the theta functions for $p = 3$. The following fact is more novel:

(46) *If P_8^3 is a set of* 8 *nodes of a quartic surface but not the set of base points of a net of quadrics* then there are only $2^6 \cdot 36$ projectively distinct sets congruent in some order to P_8^3.*

Indeed the parameters on E^4 of the 36 sets of (45) are of the type $u_1, \cdots, u_8$ or of the type $u_i - (u_1 + \cdots + u_4)/2$, $u_j + (u_1 + \cdots + u_4)/2$ $(i = 1, \cdots, 4;\ j = 5, \cdots, 8)$. If in each of these 36 sets $\sigma = \omega/2$ be added to any two or any four of the parameters all $36 \cdot 2^6$ sets of (46) are obtained. The group $e_{8,3}$ contains an invariant subgroup $i_{8,3}'$ which leaves each of the $36 \cdot 2^6$ sets unaltered. The factor group $f_{8,3}'$ has the order $8!36 \cdot 2^6$ and is isomorphic

* Cayley's *dianome* surface, *Collected Mathematical Papers*, vol. 7, p. 133.

with that group of operations on the theta functions ($p = 3$) which is generated by the transformation of the periods and by the addition of half periods to the argument.

Similar theorems can be stated for P_9^2.

(47) *A pencil of plane cubic curves can be transformed by ternary Cremona transformation into only* 960 *projectively distinct pencils of cubics. A pencil of plane sextics with* 9 *double points can be transformed into only* $2^8 \cdot 960$ *projectively distinct pencils of sextics.*

Of course these theorems are only special cases of the more general theorem that if $\sigma = \omega/r$ (r an integer) the number of projectively distinct sets congruent in some order to P_8^3 or P_9^2 is finite and equal respectively to $36N$ or $960N$, where N is the order of the finite group derived from $i_{8,3}$ or $i_{9,2}$ by the assumption $r\sigma \equiv 0$.

The remarkable utility of $e_{n,k}$ in the discussion of the special cases of $G_{n,k}$ noted above seems not to extend to other cases. The other cases however are distinguished sharply from the ones here treated by the fact that the invariant quadratic form (35) does not admit the transformation $u_i' \equiv u_i + \omega/(k+1)$.

7. The isomorphisms of $G_{n,k}$

In this paragraph some isolated facts previously noted are brought together and some of the properties of $G_{n,k}$ are summarized.

(48) *If* $G_{n,k}$ *is infinite it is simply isomorphic with* $g_{n,k}$ *and* $e_{n,k}$. *The* $G_{n,k}$ *contains an invariant subgroup* $\Gamma_{n,k}$ *of index two whose elements are products of an even number of generators. The* $G_{9,2} = G_{9,5}$ *and the* $G_{8,3}$ *have the constitution described in* (38), (39) *and* (43), (44) *of* § 6. *The invariant subgroup corresponding to* $i_{9,2}$ *is contained in* $\Gamma_{9,2}$; *that corresponding to* $i_{8,3}$ *is not completely contained in* $\Gamma_{8,3}$.

The simple isomorphism follows from the statement preceding (18) of § 4 and from (33) of § 6. The existence of $\Gamma_{n,k}$ then follows from (31) of § 5 or from (33). The element C_1 of $i_{8,3}$ has the determinant -1 while both E_9 and F_9 of $i_{9,2}$ have the determinant 1.

(49) *The* $G_{8,2} = G_{8,4}$ *is in* 1 *to* 2 *isomorphism with* $g_{8,3}$ *and* $e_{8,2}$. *It contains an invariant subgroup* $\Gamma_{8,2} = \Gamma_{8,4}$ *which is simple.*

For according to (26) § 5, $g_{8,2}$ contains an invariant involution I and this, the E_{17} of § 2, leads to the identity in $G_{8,2}$. The corresponding element of $e_{8,2}$ is $u_i' = -u_i$ ($i = 1, \cdots, 8$) of determinant 1 whence the factor group $G_{8,2}$ still has an invariant subgroup of index two. That this group is simple is proven in § 209 of Dickson's *Linear Groups*.

(50) *The* $G_{7,2} = G_{7,3}$ *is in* 1 *to* 2 *isomorphism with* $g_{7,2}$ *and* $e_{7,2}$ *and itself is a simple group.*

Here $g_{7,2}$ and $e_{7,2}$ have the same invariant involution described under (49) but in this case it has a determinant -1 so that the factor group is $\Gamma_{7,2} = G_{7,2}$. In § 3 we have seen that $G_{7,2}$ is isomorphic with the double tangent group which is known to be simple.

(51) *The $G_{6,2}$ is simply isomorphic with $g_{6,2}$ and $e_{6,2}$. The invariant subgroup $\Gamma_{6,2}$ is the known simple group of order* 25920.

(52) *The $G_{k+3,k} = G_{k+3,1} = G_{(k+3)!}$ is in 1 to 2^{k+2} isomorphism with $g_{k+3,k}$ and $e_{k+3,k}$. It has an invariant subgroup $\Gamma_{k+3,k}$ of index two which is simple when $k > 1$.*

We naturally assume that $k > 1$ to exclude the case of a P_4^1 or Q_4^1. Then $g_{k+3,k}$ contains an invariant abelian subgroup of order 2^{k+2} generated by elements which are products like that of $A_{1,\ldots,k+1}$ with the transposition $(k+2, k+3)$. All these according to § 2 lead to sets P_{k+3}^k which are projectively equivalent in the identical order whence they correspond to the identity in $G_{k+3,k}$. Also they are of determinant 1 so that $G_{k+3,k}$ still has its subgroup $\Gamma_{k+3,k}$. Since the transpositions have a determinant -1, $\Gamma_{k+3,k}$ is isomorphic with the simple alternating group $g_{\frac{1}{2}(k+3)!}$.

The above summary includes all cases in which the Cremona group $G_{n,k}$ in $\Sigma_{k(n-k-2)}$ exists though $g_{n,k}$ and $e_{n,k}$ persist for values $n = k+2, k+1$. Let us state one further theorem:

(53) *When $n = 2\nu$ or $n = 2\nu + 1$ the group $G_{n,k}$ contains ν distinct conjugate sets of involutions represented by the ν distinct types of involutions in $G_{n!}$.*

That the sets are distinct follows from the fact that in the collineation groups $e_{n,k}$ and $g_{n,k}$ they are of projectively different kinds. In case $G_{n,k}$ is finite it contains no other involutions since there are no other kinds in the finite geometry. In the infinite cases $G_{9,2}$ and $G_{8,3}$ however there exist the two new conjugate sets described in § 6, (39) and (44). The sets are distinguished by the conditions $r = 0$, $r \neq 0$. These new sets serve to generate the invariant subgroups $i_{9,2}$ and $i_{8,3}$.

Possibly the most striking feature of the above résumé is the paucity of facts concerning the general $G_{n,k}$.

8. Invariants of $G_{n,k}$

Apparently nothing can be said in a general way of the invariants of the extended group $G_{n,k}$. Each case presents features peculiar to itself and must be separately treated. The case of $G_{6,2}$ for which a complete system can be obtained is reserved for Part III of this account. For the case $G_{7,2}$ a rather interesting linear system of irrational invariants is given below from which rational invariants of $G_{7,2}$, i. e., of the ternary quartic, can be calculated. Some general conclusions as to the existence of certain invariants for the infinite cases $G_{9,2}$ and $G_{8,2}$ also are given.

Let us use for P_7 with points $p_1, \cdots, p_7$ the basis notation of § 3 according to which the degenerate cubics of the net on P_7^2 are named as follows: Q_{ij} is the line on p_i, p_j and the conic on the other five points; Q_{0i} is the point p_i and the cubic with node at p_i $(i, j = 1, \cdots, 7)$. The net of cubics on P_7^2 maps S_2 on a plane S_2' in such a way that the 28 degenerate cubics map into the 28 double tangents of a general quartic curve C^4 in S_2'. This C^4 acquires a node when either two points p coincide in some direction, or three points lie on a line, or six points lie on a conic. Hence the discriminant Δ of C^4 breaks up into 63 irrational factors which we shall denote as follows: $\delta_{ik} = 0$ is the condition that p_i and p_k coincide; $\delta_{0i} = 0$ is the condition that the six points other than p_i are on a conic; and $\delta_{0ijk} = 0$ is the condition that the points p_i, p_j, p_k are on a line.* Thus the double tangents and the discriminant factors are permuted under the operations of $G_{7,2}$ as the O quadrics and the points of S_5 are permuted under the group of the null system.

The discriminant factors do not behave alike with reference to the set P_7^2. The factor δ_{0i} is of degree two in the coördinates of each point other than p_i. The factor δ_{0ijk} is of degree one in the coördinates of p_i, p_j, p_k. But the factor δ_{ij} which vanishes if p_i and p_j coincide in some direction cannot be expressed by a single condition on the coördinates of P_7^2. When these points coincide both δ_{0ijk} and δ_{0k} must vanish. Hence the occurrence of a factor δ_{ij} in any product of discriminant factors must be indicated by the degree to which other factors vanish when $p_i = p_j$. The discriminant itself must be unaltered by permutation of the points whence it is to within a numerical factor the product of the seven squared factors δ_{0i}^2 and of the thirty-five squared factors δ_{0ijk}^2. Hence Δ is of degree 54 in the coördinates of each point and has a zero of order $20 = 2 + 18$ for each coincidence (i, j) rather than a zero of order 2 corresponding to the squared factor δ_{ij}^2. Since Δ is an invariant of C^4 of degree 27 in the coefficients the first invariant of C^4 of degree 3 in the coefficients should be of degree 6 in the coördinates of each point of P_7^2 and should have a zero of order 2 for each coincidence (i, j). This invariant is given below in (69).

It is customarily true that rational invariants can be expressed as symmetric functions of irrational invariants and the question at once arises as to whether there are irrational invariants of P_7^2 of degree 3 in the coördinates of each point which vanish at least once for each coincidence (ij). If we examine the product

$$(54) \qquad (514)(624)(235)(136)(127)(347)(567),$$

where (ijk) is the ternary determinant symbol $(p_i\, p_j\, p_k)$, we see that it is homogeneous and of degree 3 in the coördinates of each point and that it

* Cf. De Paolis, Atti di Lincei, ser. 3, vol. 1 (1877), p. 511, and vol. 2 (1878), p. 851; see also Snyder (loc. cit.).

vanishes once and only once for each coincidence. It corresponds therefore to the following product of seven factors of Δ

$$\delta_{0514}\,\delta_{0624}\,\delta_{0235}\,\delta_{0136}\,\delta_{0127}\,\delta_{0347}\,\delta_{0567}. \tag{55}$$

According to F. G., II (3), (4), the corresponding seven points in the finite geometry lie in a plane and any two are syzygetic, i. e., the seven points are in a Göpel plane. According to F. G., I (19), there are 135 such Göpel planes. The combination (54) is well known in other connections.* It is unaltered to within sign by 168 permutations of the points so that it is one of a set of 30 similar products. If we carry out on (55) the Cremona involution A_{123} the factors δ_{0514}, δ_{0624}, δ_{0347} are unaltered; the factors δ_{0235}, δ_{0136}, δ_{0127} become respectively δ_{15}, δ_{26}, δ_{37}; while the factor δ_{0567} becomes δ_{04}. Thus (54) is transformed into

$$(514)(624)(347)\Delta_4, \tag{56}$$

where $\Delta_4 = 0$ is the condition expressed in terms of the points that the points other than p_4 are on a conic. Of this product it is true also that it is homogeneous of degree 3 in the coördinates of each point, vanishes at least once for each coincidence, and vanishes twice for the coincidences $(5, 1)$, $(6, 2)$, $(3, 7)$. The corresponding product of factors of Δ is therefore

$$\delta_{0514}\,\delta_{0634}\,\delta_{0347}\,\delta_{04}\,\delta_{51}\,\delta_{62}\,\delta_{37}. \tag{57}$$

These factors again correspond to the seven points of a Göpel plane. There are 7.15 possible expressions like (56) so that the products corresponding to the 135 Göpel planes all are included in the types (54) and (56). Hence

(58) *The 63 irrational factors of the discrimination of C^4 can be grouped in 135 ways into products of the seven factors which correspond to the points of a Göpel plane. These products form a conjugate set of irrational invariants of C^4 which are permuted to within sign or to within a common outstanding factor by the operations of $G_{7,2}$.*

If in each of these *Göpel invariants* we regard one of the points say p_7 as a variable point we obtain 135 cubics on the remaining points P_6^2. An inspection of (54) and (56) shows that the cubics are those which in I, § 4, were found to be the maps of the 45 tritangent planes of the cubic surface C^3 mapped from P_6^2. It is therefore possible to express the 135 Göpel invariants in terms of $\bar{a}, \cdots, \bar{f}$ and $a, \cdots, f$ of I, § 4. This is accomplished by noting that from I (46), (47) we have

$$\begin{aligned}(531)(461)(342)(562) + (532)(462)(341)(561) &= -\overline{cf},\\ (531)(461)(342)(562) - (532)(462)(341)(561) &= -d_2,\end{aligned} \tag{59}$$

* Cf. Weber, *Algebra*, II, p. 540 (3); see also H. S. White, Proceedings of the National Academy of Sciences, vol. I, p. 464 (1915).

where $\overline{cf} = a_2 + 2(\bar{c}^2 + \bar{f}^2 + \overline{cf})$, $a_2 = \sum \bar{a}\bar{b}$, and $d_2 = \Delta_7 = 0$ is the condition that P_6^2 be on a conic. Moreover from I (29),

$$(c + f) = 4(547)(217)(367).$$

Hence by using (59) we find that

$$8(531)(461)(342)(562)(547)(217)(367)$$
$$= -(\overline{cf} + d_2)(c + f) \equiv [cf+],$$

(60) $$8(523)(462)(341)(561)(547)(217)(367)$$
$$= (\overline{cf} - d_2)(c + f) \equiv [cf-],$$
$$8\Delta_7(547)(217)(367) = 2d_2(c + f) \equiv [cf].$$

By using the parallel permutations (12), (23456); $(ad)(be)(cf)$, $(adbfe)$ of I (28), an odd permutation being accompanied by a change of sign of d_2 we get from (60) the 45 Göpel invariants which contain as factors the 15 tritangent planes of the form $c + f$. There remain to be determined the expressions for the 90 Göpel invariants of type (56) with factor other than Δ_7. In the first of equations I (38) we find

$$4\Delta_2(127) = (531)(461)(a + d) - (341)(561)(b + e).$$

If we multiply by $2(245)(236)$ so as to obtain the same degree in all the points, and if then we replace the coefficients of $(a + d)$ and $(b + e)$ by means of relations derived from (59) by permutation we get

(61) $$8\Delta_2(127)(245)(236) = -(\overline{ad} - d_2)(a + d)$$
$$+ (\overline{be} + d_2)(b + e) \equiv [ad, be],$$

which is the left member of the last equation of I (43). Had we used the factors $2(234)(256)$ or $2(235)(246)$ the remaining members of this set of equations would have obtained. Hence

(62) *The* 135 *Göpel invariants for isolated* p_7 *divide into* 45 *sets of three such that the three in a set contain as factor a tritangent plane of the cubic surface* C^3 *of* P_6^2. *These sets divide into* 15 *of the kind,* $[cf+]$, $[cf-]$, $[cf]$; *and* 30 *of the kind* $[ad, be]$, $[be, cf]$, $[cf, ad]$. *All the sets are obtained from the two given sets by transposing two letters and changing the sign of* d_2.

We can use (61) and (60) to define the sign of Δ_i in terms of the sign of d_2 as given by (59).

It is clear from (60) that the first set of three in (62) are linearly related. These three Göpel invariants correspond in the finite geometry to three Göpel planes with a common null line. According to F. G., I (19), (22), there are 315 null lines all of which are conjugate and each of which is on three Göpel planes. There must be therefore 315 such relations as follows:

$$
\begin{aligned}
&1^\circ \quad [cf] + [cf+] + [cf-] = 0,\\
&2^\circ \quad [ad, be] + [be, cf] + [cf, ad] = 0,\\
&3^\circ \quad [cf] + [be] + [ad] = 0,\\
&4^\circ \quad [ad, be] + [ad-] + [be+] = 0,\\
&5^\circ \quad [ad, be] + [be, ad] + [cf] = 0,\\
&6^\circ \quad [ab, de] + [bc, ef] + [ca, fd] = 0.
\end{aligned} \tag{63}
$$

The number of relations of types $1^\circ, \cdots, 6^\circ$ is respectively 15, 30, 15, 90, 45, 120. All of these relations are mere identities in the letters or they are satisfied due to $a + b + \cdots + f = 0$ except those of type 6°. They exist by virtue of the identity

$$
\begin{aligned}
&\overline{bc}(b+c) + \overline{ca}(c+a) + \overline{ab}(a+b)\\
&\qquad = \overline{ef}(e+f) + \overline{fd}(f+d) + \overline{de}(d+e),
\end{aligned} \tag{64}
$$

which is proved below. Hence

(65) *Corresponding to the fact that the* 135 *Göpel planes pass by threes through* 315 *null lines, the* 135 *Gopel invariants are connected by the* 315 *three-termed linear relations of* (63).

We see that the 15 invariants $[cf]$ can be expressed in terms of 5. For $[ab] + [ac] + \cdots + [af] = 8d_2\, a$ whence $d_2\, a, \cdots, d_2 f$ subject to $\sum_6 d^2 a = 0$* will serve to express all of type $[cf]$. Furthermore by adding terms like $d_2\, a$ to $[cf+]$ and $[cf-]$ these invariants are reduced to expressions of the form $\overline{cf}(c+f)$. According to (63) 4° invariants of type $[ad, be]$ can be expressed in terms of types $[cf+]$ and $[cf-]$. Thus all the Göpel invariants can be expressed by means of terms like $d_2\, a$ and $\overline{cf}(c+f)$. Terms of the latter kind are themselves linearly related as in (64). But the ten relations (64) are themselves a consequence of five linear relations. For if we symmetrize with the help of $\bar{a}$ and a we find that

$$
\overline{ab} + \overline{ac} + \cdots + \overline{af} = 6\bar{a}^2 + a_2, \tag{66}
$$

$$
\overline{ab}(a+b) + \overline{ac}(a+c) + \cdots + \overline{af}(a+f) = 2\sum_6 \bar{a}^2\, a. \tag{67}
$$

The right member of (67) is symmetric. If then we subtract the sum of relations (67) formed for d, e, f from the sum formed for a, b, c the relation (54) is obtained. Thus by equating the left members of the 6 relations (67) five independent linear relations on the 15 terms $\overline{cf}(c+f)$ are obtained and only 10 of these terms are independent. Hence

* Here and hereafter the number under Σ indicates the number of terms of the type following which are to be used in symmetrizing for six letters $a, \cdots, f$ or $\bar{a}, \cdots, \bar{f}$.

(68) The 135 *Göpel invariants lie in a linear system of irrational invariants of which only* 15 *are linearly independent. They give rise in* Σ_6 *to a linear system of spreads of order seven invariant under* $G_{7,2}$.

For if we put P_7^2 in the canonical form of I, p. 53, it is easy to see from (54) and (56) that u^2 factors out of each Göpel invariant. According to I (68) the simplest invariant linear system in Σ_6 under $G_{7,}$ has the order 8. Thus the members of this system which contain an additional factor u constitute the simplest linear system in Σ_6 invariant under the extended group $G_{7,2}$.

Any symmetric function of the squares of the 135 Göpel invariants or any unsymmetric functions of these squares whose terms are permuted by the operations of $G_{7,2}$ is a rational invariant of $G_{7,2}$ and therefore also of the allied ternary quartic C^4. The invariant of lowest degree is obtained from the sum of the squares. It is

$$I_1 = \sum_{90} [ad, be]^2 + \sum_{15} [ad +]^2 + \sum_{15} [ad -]^2 + \sum_{15} [ad]^2.$$

By making use of (63) 4° this can be written

$$I_1 = 2 \sum_{90} [ad -][be +] + 7 \sum_{15} [ad +]^2 + 7 \sum_{15} [ad -]^2 + \sum_{15} [ad]^2.$$

This reduces eventually to the following explicit expression in a and $\bar{a}$:

$$(69) \quad \begin{aligned} I_1 = 12 [\sum_{6} a^2 (8d_2^2 + 27a_3 \bar{a} - 24a_2 \bar{a}^2 - 22\bar{a}^4) \\ + 4 \sum_{15} ab\, \bar{a}\bar{b} \{4(\bar{a}^2 + \bar{b}^2) + 5\bar{a}\bar{b}\}]. \end{aligned}$$

It is for the surface C^3 a covariant quadric of degree six.

With the Göpel invariants as elements it is possible to form other conjugate sets of irrational invariants which correspond in the finite geometry to other configurations such as the O and E quadrics. Enough has been said above to show that a theory of the invariants of the quartic based on the invariants of $G_{7,2}$ may be feasible. The formulæ developed above will be useful for the problem considered in Part III of determining covariants of C^3, particularly the linear covariants.

The following remarks on the invariants of $G_{9,2}$ may serve to illustrate also the infinite case $G_{8,3}$. The set P_2^9 is on a cubic curve E^3 whose coefficients are of degree 3 in the coördinates of each point. The invariants S and T of E^3 are invariants of the extended group $G_{9,2}$. If we take P_9^2 in the canonical form of I, § 53, so that the coördinates of $p_5, p_6, \cdots, p_9$ are $y_1^{(i)}, y_2^{(i)}, u$ $(i = 1, \cdots, 5)$ then P_9^2 is represented by the point y, u of Σ_{10} and the invariants S and T give rise to invariant spreads in Σ_{10} of orders $48 - \lambda$ and $72 - \mu$. Here λ, μ are the number of additional factors u which occur in the S and T of E^3 formed for the canonical form of P_9^2—numbers which could

be determined by a somewhat tedious calculation. Hence in Σ_{10} $G_{9,2}$ has the pencil of invariant spreads $S^3 + kT^2 = 0$. The base of this pencil is a manifold M_8 determined by $S = T = 0$, which breaks up into two parts M_8' and M_8''. The one part M_8' is the map of sets P_9^2 for which E^3 has a cusp. The other part M_8'' is the map of sets P_9^2 for which E^3 vanishes, i. e., sets which are the base points of a pencil of E^3's. Both of these *manifolds M_8' and M_8'' are rational.* For in the case of M_8' the P_9^2 can be put in the form $x_{i1} = t_i^3$, $x_{i,2} = t_i$, $x_{i,3} = 1$ $(i = 1, \cdots, 9)$. The transformation $t' = \alpha t$ $(\alpha \neq 0)$ gives rise to a projectively equivalent P_9^2 so that only the ratios of the 9 parameters are essential. If the above set P_9^2 be transformed linearly into the canonical form, the coördinates y, u appear as functions of the 9 homogeneous parameters t. In the case of M_8'' the same result follows since the coördinates of the 9th base point of a pencil of cubics are rational functions of the coördinates of the other 8 base points. Though both of the manifolds M_8' and M_8'' are invariant under $G_{9,2}$ their behavior is quite different. *On the invariant manifold M_8'' the infinite group $G_{9,2}$ effects only a finite number of distinct transformations* which constitute a group of order 8!8640. In fact this group is merely a representation of the factor group of the invariant subgroup $i_{9,2}$ of $G_{9,2}$ described in § 6. *On the invariant manifold M_8' the group $G_{9,2}$ effects transformations which are represented on the parameters $t_1, \cdots, t_9$ by the operations of $e_{9,2}$ of* § 6. To prove this one has only to test the effect on the parameters t of the transformation A_{123} and note that it is the same as (32) of § 6.

The continuous pencil of invariant spreads $S^3 + kT^2 = 0$ is cut by an infinite but discontinuous system of invariant algebraic spreads $M_9(r)$, r any positive integer. The spread $M_9(r)$ is the map of sets P_9^2 for which the sum of the elliptic parameters on E^3 is a primitive rth period. The degree of this invariant $M_9(r)$ in the coördinates of each point of P_9^2 is 3ν where ν is the number of primitive rth periods. For if $p_1, \cdots, p_8$ be given each member of the pencil of cubics on P_8^2 contains ν points each of which makes up with P_8^2 a P_9^2 of the required type. The ν points run over a curve $C(r)$ which must be of order $3k$ with k-fold points at $p_1, \cdots, p_8$ since its characteristic property is invariant under $G_{8,2}$. Hence $C(r)$ meets an E^3 of the pencil in $9k - 8k = \nu$ points apart from P_8^2. The order of $M_9(r)$ in Σ_{10} is $15\nu - \lambda$ if u^λ factors out of $M_9(r)$ for the canonical form of P_9^2. On each of these invariant spreads $G_{9,2}$ effects only a finite number of distinct transformations—a number which could readily be determined for any particular value of r.

BALTIMORE
December 1, 1915

INFINITE PRODUCTS OF ANALYTIC MATRICES

BY

GEORGE D. BIRKHOFF

In a large part of the theory of functions of a single complex variable the matrix of analytic functions rather than the single analytic function must be taken as the fundamental element. This is certainly the case for the functions defined by linear difference and differential equations.

The goal of the present paper is to show that the classical results of Weierstrass and Mittag-Leffler, treating of the formation of infinite products of functions with assigned singularities, admits of a natural extension to infinite products of matrices. The matrices considered will be square matrices of n^2 elements and of determinant not identically zero.

The concept of equivalence, which I have developed elsewhere,† lies at the basis of this extension: Let

$$A(x) = [a_{ij}(x)] \quad \text{and} \quad B(x) = [b_{ij}(x)] \quad (i, j = 1, \cdots, n),$$

be two matrices of analytic functions, each analytic in the neighborhood of a point $x = x_0$ but not necessarily analytic at the point x_0. If the matrix $M(x)$ defined by the matrix equation

$$A(x) = M(x)B(x)$$

is composed of elements $m_{ij}(x)$ each analytic at $x = x_0$ and if the determinant $|M(x)|$ of the matrix $M(x)$ is not zero at x_0, then $A(x)$ is *equivalent* to $B(x)$ at $x = x_0$ (from the left).‡

From this definition it follows immediately that if $A(x)$ is equivalent to $B(x)$ at $x = x_0$, then $B(x)$ is also equivalent to $A(x)$; if, further, $B(x)$ is equivalent to $C(x)$, then $A(x)$ is equivalent to $C(x)$.

Two matrices equivalent to each other at a point have essentially the same type of singularity at the point. It is this fact which gives importance to the notion.

Evidently any two matrices of functions analytic at x_0 are equivalent if the determinants of the two matrices do not vanish at the point. For this reason a point for which the elements of a matrix are analytic, and the determinant

* Presented to the Society, December 28, 1915.

† See, for example, Proceedings of the American Academy of Arts and Sciences, vol. 49 (1913), pp. 521–568; in particular p. 540.

‡ For the elements of the theory of matrices assumed in the present paper see Schlesinger, *Vorlesungen über linearen Differentialgleichungen*, pp. 18–19.

of the matrix is not zero, will be called an *ordinary point* of the matrix. If the elements of the matrix are analytic, while the determinant vanishes, the matrix will be said to have a *singular point of analytic type* at $x = x_0$; if the determinant vanishes to the kth order, this singular point will be said to be of *multiplicity* k.

If the elements have poles of the kth order at most at $x = x_0$, the matrix will be said to have a *pole of the kth order*; at such a point the determinant of the matrix will be analytic or have a pole of order not greater than nk.

Finally, if the elements are analytic in the neighborhood of a given point while some elements have an essential singularity at the point, and if the determinant of the matrix does not vanish in the neighborhood of the point, the matrix will be said to have an *isolated essentially singular point* there; if, however, the determinant does vanish in every neighborhood of the point, or if the point under consideration is a cluster point of singular points of the matrix not of analytic type, the matrix will be said to have a *non-isolated essentially singular point* there.

It is to be recalled that in the theory of the factorization of an analytic function a cluster point of zeros does not play the rôle of an *isolated* essentially singular point.

1. Factorization of polynomial matrices

We will commence with the consideration of matrices whose elements are polynomials. By the *order* of such a matrix we shall mean the degree of its polynomial determinant in x.

From the standpoint of factorization the simplest type is that of order zero (constant determinant). The inverse of such a matrix and the product of two such are also of order zero. These matrices are not singular in the finite plane.*

A matrix of polynomials will have a single singular point of analytic type in the finite plane if its determinant is of the form $c(x - x_0)^k$. This point will lie at $x = x_0$ and be of multiplicity k.

An *elementary* matrix will be a matrix of the form

$$U(x) = \begin{bmatrix} 1 & 0 & \cdots & c_1 & \cdots & 0 \\ 0 & 1 & \cdots & c_2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & (x - x_0) & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & c_n & \cdots & 1 \end{bmatrix} \tag{1}$$

* Questions of factorization for such matrices have been considered by Hensel and Landsberg, *Theorie der algebraischen Funktionen einer Variabeln*, etc., p. 163.

and differs from the unit matrix only in some kth column where the successive elements are the following:

$$c_1, c_2, \cdots, c_{k-1}, x - x_0, c_{k+1}, \cdots, c_n.$$

Its only finite singular point will be one of analytic type and multiplicity 1 at $x = x_0$. The kth column of the matrix $U(x)$ will be termed its *characteristic* column.

LEMMA. *The most general matrix $F(x)$ with a singular point of analytic type at $x = x_0$ of multiplicity k may be decomposed into the product of a matrix $Q(x)$, for which x_0 is an ordinary point if $k = 1$ or a singular point of analytic type and of multiplicity $k - 1$ if $k > 1$, and a second elementary matrix $U(x)$ with singular point at x_0, i. e.,*

$$F(x) = Q(x)\,U(x). \tag{2}$$

Proof.—The determinant of $F(x_0)$ is zero by hypothesis and thus we may determine a matrix of constants D of determinant not zero so that the product matrix $F(x_0)\,D$ has the elements of some lth column all zero. To effect this it is necessary merely to determine the elements $d_{1l}, d_{2l}, \cdots, d_{nl}$ (not all zero) of that column of D so that the n equations

$$f_{i1}(x_0)\,d_{1l} + f_{i2}(x_0)\,d_{2l} + \cdots + f_{in}(x_0)\,d_{nl} = 0 \quad (i = 1, 2, \cdots, n)$$

hold, and then to choose the remaining elements of D so that its deteminant does not vanish. But this set of n linear homogeneous equations in d_{1l}, $d_{2l}, \cdots, d_{nl}$ can always be satisfied since the determinant of the coefficients is $|F(x_0)| = 0$.

Furthermore let us assume for the moment that the first of these quantities d_{1l} is not zero and let us take $l = 1$. In this case the elements of D occurring in the $n - 1$ columns after the first may be taken to be 1 along the principal diagonal and zero elsewhere. Also the elements of the first column may be taken to be $1, d_2, d_3, \cdots, d_n$ so that

$$D = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ d_2 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ d_n & 0 & \cdots & 1 \end{bmatrix}.$$

When D has been taken in this way, the matrix $F(x)\,D$ has the property that the elements of the first column vanish at $x = x_0$. In consequence, if we write

$$F(x)\,D = Q(x) \begin{bmatrix} x - x_0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 1 \end{bmatrix},$$

so that $Q(x)$ is derived from $F(x)D$ by dividing the elements of the first column of $F(x)D$ by $x - x_0$, the matrix $Q(x)$ is made up of elements analytic at x_0.

Moreover, if $k = 1$, the determinant of $Q(x)$ is not zero at $x = x_0$; this follows from the determinant relation $|F(x)| = |Q(x)|(x - x_0)$. Likewise, if $k > 1$, the same relation shows that $Q(x)$ has a singular point of analytic type of multiplicity $k - 1$.

If we multiply the above matrix equation through by D^{-1} on the right, we obtain

$$F(x) = Q(x) \begin{bmatrix} x - x_0 & 0 & \cdots & 0 \\ -d_2 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ -d_n & 0 & \cdots & 0 \end{bmatrix}.$$

The second factor is evidently an elementary matrix $U(x)$ while the first factor $Q(x)$ has the properties demanded of it in the lemma. The first column of $U(x)$ is the characteristic column.

In the case when d_{1l} is zero some one of the quantities $d_{2l}, \cdots, d_{nl}$ will be different from zero. If d_{2l} is not zero, for example, we may take $l = 2$ and factor $F(x)$ into the product of a matrix $Q(x)$ and an elementary matrix $U(x)$ whose second column is its characteristic column. In general if the λth of the quantities $d_{1l}, \cdots, d_{nl}$ is not zero we may take $l = \lambda$ and proceed as before.

This completes our proof of the lemma.

It is evident, in the simple case when $F(x)$ has a singular point of multiplicity one at $x = x_0$, that there may be n distinct ways of separating out an elementary factor, namely by allowing l to range from 1 to n. Hence the theory of factorization for $n > 1$ diverges greatly from that for the case $n = 1$ inasmuch as the mode of factorization is not unique.

The matrix $Q(x)$ of the lemma has the same singular points in the finite plane for $x \neq x_0$ as $P(x)$. Furthermore such a singularity is of the same kind for both matrices, i. e., is of analytic type with the same multiplicity, or a pole of order k, or an isolated essential singularity, or a non-isolated essential singularity for both. This is an obvious consequence of (2) and the inverse relation $Q(x) = F(x)U^{-1}(x)$.

We are now in a position to establish at once the following theorem:

THEOREM I. *An arbitrary matrix of polynomials $P(x)$ of order m whose determinant is factorable into the product*

$$c(x - x_1)(x - x_2) \cdots (x - x_m) \qquad (c \neq 0)$$

may be expressed as a product

$$P_0(x)\,U_1(x)\,U_2(x)\cdots U_m(x)$$

where $P_0(x)$ is a polynomial matrix of order zero and where $U_1(x)$, $U_2(x)$, $\cdots$, $U_m(x)$ are elementary matrices with singular points at x_1, x_2, $\cdots$, x_m respectively.

It is obvious that a successive application of the above lemma enables us to write $P(x)$ in the stated form.

Suppose now that a polynomial matrix $P(x)$ be expressed in any way as a product $V_1(x)V_2(x)\cdots V_m(x)$ of m polynomial matrices $V_1(x)$, $V_2(x)$, $\cdots$, $V_m(x)$, each having its only finite singularity at distinct points x_1, x_2, $\cdots$, x_m respectively. A particular product of this sort can be obtained at once from the product for $P(x)$ of Theorem I.

The most general factored form $\overline{V}_1(x)\overline{V}_2(x)\cdots\overline{V}_m(x)$ of $P(x)$ with the same properties is defined by the equations

$$\begin{aligned}
\overline{V}_1(x) &= V_1(x)R_1(x),\\
\overline{V}_2(x) &= R_1^{-1}(x)V_2(x)R_2(x),\\
&\cdots\cdots\cdots\cdots\\
\overline{V}_m(x) &= R_{m-1}^{-1}(x)V_m(x),
\end{aligned}$$

where $R_1(x)$, $\cdots$, $R_{m-1}(x)$ are arbitrary polynomial matrices of order zero.

It is clear that the indicated general factored form is a possible form.

Further, any factorization $\overline{V}_1(x)\overline{V}_2(x)\cdots\overline{V}_m(x)$ leads at once to the matrix identity

$$V_2(x)\cdots V_m(x)\overline{V}_m^{-1}(x)\cdots\overline{V}_2^{-1}(x) = V_1^{-1}(x)\overline{V}_1(x).$$

The matrix $R_1(x)$, as given by the right-hand member, has its only possible finite singularity at x_1, while, as given by the left-hand member, it has its only possible singularities at $x_2, \cdots, x_m$. Hence $R_1(x)$ is not singular in the finite plane and must be composed of polynomial elements. This gives us the first equation.

By use of the relation just obtained we find

$$V_2(x)\cdots V_m(x) = R_1(x)\overline{V}_2(x)\cdots\overline{V}_m(x),$$

and from this we get further

$$V_3(x)\cdots V_m(x)\overline{V}_m^{-1}(x)\cdots V_3^{-1}(x) = V_2^{-1}(x)R_1(x)\overline{V}_2(x).$$

The matrix given by either side of this equation must, for a similar reason, be a polynomial matrix $R_2(x)$ of order zero. This conclusion leads us at once to the second of the given equations.

Continuing in this way we derive all of the desired equations in succession.

It is interesting to consider a particular resolution in elementary factors

possible in general when the elements of $P(x)$ are polynomials of the μth degree and when $|P(x)|$ is a polynomial of degree $n\mu = m$ with distinct roots $x_1, x_2, \cdots, x_m$.

If we remove an elementary factor corresponding to $x = x_m$ for which the first column is the characteristic column, we thereby only alter the first column of $P(x)$. In fact we obtain the quotient $Q(x)$ of the lemma from $P(x)$ by multiplying on the right by $U^{-1}(x)$, which is equivalent to adding a constant multiple of the second, third, $\cdots$, last column to the first column, and dividing that column by $x - x_m$. Thus the degree of the first column is lowered by unity, while the other columns are unaltered.

In the same way we may lower the degree of the second column by unity, by the removal of an elementary factor corresponding to $x = x_{m-1}$ for which the second column is the characteristic column.

By this process, after n steps, we lower the degree of all n columns of $P(x)$ by unity by removing n elementary factors.

Repeating the process we can reduce the quotient to a constant matrix in $n\mu$ steps. Every step can be carried out in the *general* case, i. e., if certain particular algebraic relations between the coefficients of $P(x)$ do not hold.

Thus the matrix $P_0(x)$ of Theorem I of order zero can be taken to be a matrix of constants in this case.

This result shows that one can not expect to obtain a simpler normal form for a factor with one singular point than the elementary matrix. For we have used the $n\mu$ roots $x_1, \cdots, x_m$, the $n\mu$ elementary factors each involving $n - 1$ arbitrary constants, and a constant matrix with n^2 constants. Thus there are involved $n^2(\mu + 1)$ explicit constants in the factorization of $P(x)$, precisely as many as there are in the polynomial matrix $P(x)$. No further simplification can be expected since no superfluous constants have been employed.

2. Equivalence problem for polynomial matrices

In the first paragraph we solved the simple problem of factorization of polynomial matrices. The converse problem of constructing a polynomial matrix with given factors is solved merely by writing down the product of the factors.

But it is possible to modify the point of view slightly. Every polynomial matrix with singularities at the distinct points $x_1, \cdots, x_m$ is by the lemma of § 1 equivalent to a polynomial matrix $V_i(x)$ at the point x_i whose only finite singularity lies at x_i for $i = 1, 2, \cdots, m$.

We are thus led to formulate a converse problem which may be stated as follows:

Equivalence problem. *Given polynomial matrices $V_1(x), \cdots, V_m(x)$, each having its only finite singular point at the distinct points $x_1, \cdots, x_m$ re-*

spectively; to construct a polynomial matrix $P(x)$ equivalent to these matrices at $x_1, \cdots, x_m$ respectively but with no finite singular points other than $x_1, \cdots, x_m$.

Before proceeding to its solution we may note that the formulation of the problem is symmetrical with respect to all the singular points involved.

An explicit solution may be obtained by the aid of a simple notation. Let $[T(x)]_\xi$ denote a polynomial matrix, with its only finite singularity at ξ, to which the polynomial matrix $T(x)$ is equivalent at the singular point $x = \xi$ of $T(x)$. The lemma of factorization of § 1 shows that, since $T(x)$ is composed of elements analytic at $x = \xi$, a matrix $[T(x)]_\xi$ given as a product of elementary matrices, will always exist. If now we write for brevity

$$
\begin{aligned}
P_1(x) &= [V_1(x)]_{x_1} = V_1(x), \\
P_2(x) &= [V_2(x) P_1^{-1}(x)]_{x_2}, \\
&\cdots\cdots\cdots\cdots \\
P_m(x) &= [V_m(x) P_1^{-1}(x) \cdots P_{m-1}^{-1}(x)]_{x_m},
\end{aligned} \tag{3}
$$

a solution is

$$
P(x) = P_m(x) P_{m-1}(x) \cdots P_1(x). \tag{4}
$$

To prove that $P(x)$, as defined in this manner, has the desired properties we begin by observing that $P(x)$ is certainly equivalent to $V_1(x)$ at $x = x_1$ since the product $P_m(x) \cdots P_2(x)$ is not singular at $x = x_1$.

Next let us proceed to the point x_2. By definition of $P_2(x)$ we may write

$$
P_2(x) = A(x) V_2(x) P_1^{-1}(x)
$$

where $A(x)$ is not singular at $x = x_2$. Consequently, on substituting in this value for $P_2(x)$, we see that

$$
P(x) = P_m(x) \cdots P_3(x) A(x) V_2(x).
$$

But $P_m(x), \cdots, P_3(x)$ and $A(x)$ are not singular at $x = x_2$ so that $P(x)$ is evidently equivalent to $V_2(x)$ at $x = x_2$.

In precisely the same way we proceed with the proof that $P(x)$ is equivalent to $V_3(x), \cdots, V_m(x)$ at $x_3, \cdots, x_m$ respectively, and thus forms a solution.

The most general solution $\overline{P}(x)$ of the equivalence problem is $P_0(x) P(x)$ where $P_0(x)$ is any polynomial matrix of order zero.

In the first place it is clear that $\overline{P}(x)$ as so defined is a solution of the equivalence problem. Moreover, consider any solution $\tilde{P}(x)$. At any singular point as x_1 we have

$$
\begin{aligned}
P(x) &= A(x) V_1(x), \\
\tilde{P}(x) &= B(x) V_1(x),
\end{aligned}
$$

where $A(x)$ and $B(x)$ are not singular at $x = x_1$. Using these values of

P and $\tilde{P}$ we conclude that the matrix $\tilde{P}(x)[P(x)]^{-1}$ is equal to $B(x)A^{-1}(x)$ and thus is not singular at x_1. Similarly this matrix is not singular at x_2, $\cdots$, x_m and also of course at any other point of the finite x-plane. Since $\tilde{P}(x)$ by hypothesis is a polynomial matrix the matrix $P_0(x)$ under consideration must be a polynomial matrix of order zero, which we were to prove.

3. Factorization of matrices of entire functions. First method

Let $E(x)$ be a matrix of entire functions. Its finite singularities are all of analytic type, and fall at the points $x_1, x_2, \cdots$, at which the determinant of $E(x)$ vanishes. We take these points to be arranged in some arbitrary order and assume that they are infinitely many in number. It is clear that we have $\lim x_m = \infty$.

Suppose that $V_1(x)$ is some polynomial matrix with unique finite singularity at x_1, to which $E(x)$ is equivalent at x_1; for instance $V_1(x)$ may be a product of elementary matrices with singularity at x_1, as follows from the lemma of § 1. The matrix $E(x)V_1^{-1}(x)$ is a matrix of entire functions without a singularity at $x = x_1$ but with its other finite singularities at $x_2, x_3, \cdots$. At x_2 this modified matrix is equivalent to a polynomial matrix $V_2(x)$ with unique finite singularity at x_2. Forming now the matrix of entire functions $E(x)V_1^{-1}(x)V_2^{-1}(x)$, which is not singular at x_1 and x_2, we are led to $V_3(x)$, $V_4(x), \cdots$ in similar fashion.

It is therefore possible to find such polynomial matrices $V_1(x), V_2(x), \cdots$ that $E(x)$ is equivalent to $V_1(x)$ at x_1, to $V_2(x)V_1(x)$ at x_2, and so on. The following theorem shows that matrices $E(x)$ of entire functions having this property exist for any choice of $V_1(x), V_2(x), \cdots$ with a single finite singularity of analytic type at $x_1, x_2, \cdots$ respectively.

Theorem II. *If $V_1(x), V_2(x), \cdots$ are polynomial matrices each with a single finite singularity of analytic type at the distinct points $x_1, x_2, \cdots$ and if $\lim x_m = \infty$, there exists a matrix of entire functions equivalent to the product $V_m(x) \cdots V_1(x)$ at x_m $(m = 1, 2, \cdots)$ and of the form*

$$E(x) = \lim_{m=\infty} Q_m(x)V_m(x) \cdots V_1(x), \tag{5}$$

where $Q_m(x)$ $(m = 1, 2, \cdots)$ is a matrix of entire functions without singularities in the finite plane.

Proof. Let us first attempt to choose $Q_1(x), Q_2(x), \cdots$ so that the sequence of matrices

$$R_m(x) = Q_m(x)V_m(x) \cdots V_1(x)$$

converges to a limit matrix as m approaches infinity. Now we have clearly

$$R_{m+1}(x) = [I + S_m(x)][I + S_{m-1}(x)] \cdots [I + S_1(x)]Q_1(x), \tag{6}$$

where $I + S_m(x) = Q_{m+1}(x)V_{m+1}(x)Q_m^{-1}(x)$.

It will therefore suffice to establish that the matrices $S_m(x)$ can be made to have elements of as small absolute value as desired within a circle of radius less than x_m with center at the origin. This will be established by means of the following lemma:

LEMMA. *If $F(x)$ be a matrix without singularity in some closed continuum C in the finite plane, it is possible to find a matrix $F_1(x)$ without singularity in the finite plane and such that the absolute values of the elements of the difference matrix $F(x) - F_1(x)$ are arbitrarily small throughout C.*

A very simple proof of the lemma may be given as follows. The elements of the matrix

$$\frac{dF(x)}{dx} F^{-1}(x) = A(x)$$

are not singular in C. The elements $a_{ij}(x)$ of $A(x)$ may then be represented by series of polynomials which converge uniformly throughout C, by a well-known theorem due to Runge.* Hence we may break off the terms of these series at such a place that the elements of the resulting matrix $A_1(x)$ differ from those of $A(x)$ by a uniformly small quantity throughout C.

Consider now the matrix differential equation

$$\frac{dF_1(x)}{dx} = A_1(x) F_1(x).$$

By comparison with the equation for F,

$$\frac{dF(x)}{dx} = A(x) F(x),$$

we infer that a solution $F_1(x)$ of the preceding equation may be found whose elements are analytic and differ from those of $F(x)$ by as little as we please throughout the given continuum C.

But the differential equation for F_1 is without finite singular points so that $|F_1(x)|$ nowhere vanishes.† Hence the matrix $F_1(x)$ has the desired properties.

Let us now return to the proof of the theorem. Suppose that the matrices $Q_1, \cdots, Q_m$ have been so selected that the matrices $S_1, \cdots, S_{m-1}$ determined by this choice have their elements in absolute value less than assigned positive quantities $a_1, \cdots, a_{m-1}$ respectively within given circles of radii less than $|x_2|, \cdots, |x_m|$. We shall now show that Q_{m+1} can then be so selected that the corresponding matrix S_m has its elements less in absolute value than a prescribed positive quantity a_m within any given circle of radius less than $|x_{m+1}|$.

*Acta Mathematica, vol. 6 (1884), p. 229.

† Schlesinger, loc. cit., p. 23.

To this end we observe that the matrix

$$Q_m(x)\,V_{m+1}^{-1}(x)$$

will not be singular except at $x = x_{m+1}$. Thus, by the lemma above, a matrix $Q_{m+1}(x)$ without finite singularities may be chosen so that the difference matrix

$$Q_{m+1}(x) - Q_m(x)\,V_{m+1}^{-1}(x)$$

has elements of as small absolute value as desired for $|x| \leqq d_m < |x_{m+1}|$. Hence the same is true of

$$S_m(x) = Q_{m+1}(x)\,V_{m+1}(x)\,Q_m^{-1}(x) - I,$$

as we see by multiplying the preceding expression on the right by $V_{m+1}(x)Q_m^{-1}(x)$.

Thus we have a method of assigning $Q_{m+1}(x)$ as soon as $Q_m(x)$ has been chosen, in such a way that the elements of S_m are less than a positive constant a_m for $|x| < d_m < |x_{m+1}|$, where $a_1 + a_2 + \cdots$ is a convergent series. Moreover, since $\lim x_m = \infty$ we may take $\lim d_m = \infty$ also.

But the elements of the infinite sequence of matrix products

$$R_{m+1}(x)\,Q_1^{-1}(x) = [I + S_m(x)]\,[I + S_{m-1}(x)] \cdots [I + S_1(x)] \qquad (m = 1, 2, \cdots)$$

will certainly converge absolutely and uniformly to the elements of a limit matrix if the elements of $S_m(x)$ are less in absolute value than the mth term of a convergent series of positive terms.* Furthermore the determinant of this limit matrix is the limit of the product of the determinants of the factors and the sum of the logarithms of these factors will form a convergent series. Hence we infer that $\lim R_m(x)$ is a matrix of entire functions $E(x)$ of the form (5) and of determinant not zero save at $x_1, x_2, \cdots$ at which points $E(x)$ is equivalent to $V_1(x)$, $V_2(x)V_1(x)$, $\cdots$ respectively, so that $E(x)$ has the desired properties.

The most general matrix $E(x)$ of entire functions with its singularities at $x_1, x_2, \cdots$ and equivalent to $V_1(x)$, $V_2(x)V_1(x)$, $\cdots$ at these points respectively has the form $Q(x)\,E(x)$ where $Q(x)$ is a matrix of entire functions with no finite singular point. This fact is immediately evident.

In the case $n = 1$ the matrix $Q_m(x)$, as determined by the method of the lemma, is merely an exponential with an exponent which is polynomial in x. Thus we have a thoroughgoing generalization of the factorization theorem in the case $n = 1$, but one which appears much less explicit. Furthermore, the determination of each convergence factor $Q_{m+1}(x)$ is made to depend on the previous factor $Q_m(x)$.

* The proof for $n > 1$ is not essentially more difficult than in the familiar case $n = 1$.

4. FACTORIZATION OF MATRICES OF ENTIRE FUNCTIONS. SECOND METHOD

A more explicit form of factorization can be given by the aid of an *extended elementary matrix*

$$U(x) = \begin{bmatrix} 1 & 0 & \cdots & \left[c_1\left(\frac{x}{x_0}\right)^r\right] & \cdots & 0 \\ 0 & 1 & \cdots & \left[c_2\left(\frac{x}{x_0}\right)^r\right] & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdots & \cdots & \cdot \\ 0 & 0 & \cdots & e^{p_r(x/x_0)}\left(1-\frac{x}{x_0}\right) & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdots & \cdots & \cdot \\ 0 & 0 & \cdots & \left[c_n\left(\frac{x}{x_0}\right)^r\right] & \cdots & 1 \end{bmatrix}, \tag{7}$$

where

$$p_r(x) = x + \frac{x^2}{2} + \frac{x^3}{3} \cdots + \frac{x^r}{r}. \tag{8}$$

This matrix differs from the unit matrix only in some lth column and reduces essentially to an elementary matrix for $r = 0$ (compare with (1)).

It is apparent that for a given value of r the lemma of § 1 may be extended to this type of matrix; i. e., a matrix $F(x)$ with singularity of analytic type at $x = x_0$ of multiplicity k may be written as $Q(x)U(x)$ where Q has its elements analytic at x_0, and has a singularity of analytic type of order $k-1$ at $x = x_0$ if $k > 1$, or is not singular at x_0 if $k = 1$.

In fact, if we multiply $F(x)$ on the right by a matrix

$$D(x) = \begin{bmatrix} 1 & 0 & \cdots & d_1\left(\frac{x}{x_0}\right)^r & \cdots & 0 \\ 0 & 1 & \cdots & d_2\left(\frac{x}{x_0}\right)^r & \cdots & 0 \\ 0 & 0 & \cdots & 1 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdots & \cdots & \cdot \\ 0 & 0 & \cdots & d_n\left(\frac{x}{x_0}\right)^r & \cdots & 1 \end{bmatrix},$$

which takes the same form as D of § 1 for $x = x_0$ and in which the lth column contains the d's, we may reduce the elements of the lth column of $F(x)$ to zero at $x = x_0$ (for some l). Consequently if we write

$$F(x)D(x) = Q(x)\begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdots & \cdots & \cdot \\ 0 & 0 & \cdots & e^{p_r(x/x_0)}\left(1-\frac{x}{x_0}\right) & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdots & \cdots & \cdot \\ 0 & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}$$

the elements of $Q(x)$ have the desired properties, since $Q(x)$ is derived from $F(x)D(x)$ merely by dividing the lth column by

$$e^{p_r(x/x_0)}\left(1-\frac{x}{x_0}\right).$$

Multiplying through the above equation by the inverse of $D(x)$ we have $F(x)$ expressed as the product of $Q(x)$ and an extended elementary matrix $U(x)$ like (7) with $c_1 = -d_1$, $c_2 = -d_2$, $\cdots$, $c_n = -d_n$.

It is to be especially observed that the choice of the quantities d_i and accordingly c_i is independent of the choice of r which does not enter explicitly into $D(x)$ for $x = x_0$.

We are now prepared to demonstrate immediately the following second factorization theorem for matrices of entire functions:

THEOREM III. *Any matrix of entire functions $E(x)$ with its singularities at the distinct points $x_1, x_2, \cdots$ may be written in the form*

$$E_0(x)\lim_{m=\infty} U_m(x)U_{m-1}(x)\cdots U_1(x)$$

where $E_0(x)$ is a matrix of entire functions of determinant nowhere zero and where $U_1(x), U_2(x), \cdots$ are extended elementary matrices of the form (7) such that the finite singular point of $U_1(x), U_2(x), \cdots, U_k(x)$ lies at x_1, that of $U_{k+1}(x), \cdots U_l(x)$ lies at x_2, and so on.

Proof. The possibility of selecting extended elementary matrices $U_1(x), \cdots, U_k(x)$ with singular point at x_1, such that the product $U_k(x)\cdots U_1(x)$ is equivalent to $E(x)$ at $x = x_1$, is immediately obvious. In fact we may determine the product by successive application of the extended form of the lemma of § 1, so that it is equivalent to $E(x)$ at x_1. Consider now the product

$$E(x)[U_k(x)\cdots U_1(x)]^{-1},$$

which is a matrix of entire functions with its finite singular points at x_2, $x_3, \cdots$. Treating this matrix at x_2 as we did $E(x)$ at x_1, we can find a second product of the form $U_{k+1}(x)\cdots U_l(x)$ to which the product matrix under

consideration is equivalent at x_2. Proceeding in this way we obtain a sequence of matrices $U_1(x), U_2(x), \cdots$.

We can, in addition, secure the uniform convergence of the elements of the product matrix

$$U_m(x)\,U_{m-1}(x)\cdots U_1(x),$$

for $\lim m = \infty$, within any arbitrary circle $|x| = \rho$ in the complex plane.

For we have noted that, at each step in the determination of the successive factors, the constants $c_1, c_2, \cdots, c_n$ are chosen independently of the integer r. Thus, at each step after these constants have been chosen, we can take r so large that the elements of the lth column of that factor,

$$c_1\left(\frac{x}{x_k}\right)^r,\ c_2\left(\frac{x}{x_k}\right)^r,\ \cdots,\ e^{p_r(x/x_k)}\left(1-\frac{x}{x_k}\right),\ \cdots,\ c_n\left(\frac{x}{x_k}\right)^r,$$

differ from the corresponding elements of the set

$$0,\ 0,\ \cdots,\ 1,\ \cdots 0$$

within any assigned circle $|x| = \rho < |x_k|$ by an arbitrarily small amount. Hence the elements of the corresponding factor $U_m(x)$ can be made to differ from the elements of the unit matrix by arbitrarily small quantities throughout this circle.

It is therefore apparent that the factors $U_1(x), U_2(x), \cdots$ can be so taken that the limit of their product exists uniformly within every fixed circle with center at the origin in the complex plane. The elements of this limit matrix $E_1(x)$ are of course entire functions, and it has the desired equivalence properties at $x_1, x_2, \cdots$. The matrix $E_0(x)$ defined by the equation

$$E(x) = E_0(x)\,E_1(x)$$

is a matrix not singular in the finite plane.

5. Equivalence problem for matrices of entire functions

The results of § 3 permit us to solve the following problem:

Equivalence problem. *Given polynomial matrices $V_1(x), V_2(x), \cdots$ each with its only finite singular point at the distinct points $x_1, x_2, \cdots$ respectively; to construct a matrix $E(x)$ of entire functions equivalent to $V_1(x), V_2(x), \cdots$ at $x_1, x_2, \cdots$ respectively but with no other finite singular points.*

In fact, let us use the bracket notation $[T(x)]_\xi$ as before to denote some polynomial matrix equivalent to $T(x)$ at $x = \xi$ but otherwise not singular in the finite plane. Let us write

$$\overline{V}_1(x) = [V_1(x)]_{x_1} = V_1(x),$$
$$\overline{V}_2(x) = [V_2(x)\overline{V}_1^{-1}(x)]_{x_2},$$
$$\overline{V}_3(x) = [V_3(x)\overline{V}_1^{-1}(x)\overline{V}_2^{-1}(x)]_{x_3},$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot.$$

Consider now the matrix of entire functions

$$E(x) = \lim_{m=\infty} Q_m(x)\overline{V}_m(x)\cdots\overline{V}_1(x),$$

where the matrices $Q_m(x)$ are chosen as in Theorem II so that the limit is approached uniformly. It is readily seen that the product $\overline{V}_m(x)\cdots\overline{V}_1(x)$ is equivalent to $V_i(x)$ at $x = x_i$ for $i = 1, 2, \cdots, m$. This is obvious for $i = 1$. Furthermore at $x = x_2$ we may write

$$\overline{V}_2(x) = A(x)V_2(x)\overline{V}_1^{-1}(x)$$

where $A(x)$ is not singular at $x = x_2$. It follows at once that $\overline{V}_2(x)\overline{V}_1(x)$ and hence $\overline{V}_m(x)\cdots\overline{V}_1(x)$ is equivalent to $V_2(x)$ at $x = x_2$. Continuing in this way we may establish the truth of the statement made for $i = 1, 2, \cdots, m$.

The definition of $E(x)$ now shows that it has the stated properties. Thus the equivalence problem has been solved.

The most general solution of the equivalence problem is $E_0(x)E(x)$ where $E_0(x)$ is an arbitrary matrix of entire functions of determinant not zero in the finite plane.

6. Matrices with a finite number of essential singularities

Thus far we have restricted attention to matrices with singular points of analytic type. We propose now to consider the question of the factorization of matrices with isolated essential singularities. We will restrict attention at first to the case in which there are only a finite number of isolated essentially singular points.

I shall assume the truth of the following lemma:

Lemma. *Let $L(x)$ be a matrix of functions for which $x = \infty$ is an isolated essentially singular point. Then it is possible to write $L(x)$ in the product form $A(x)E(x)(x^{k_j}\delta_{ij})$ where $A(x)$ is a matrix of functions not singular at $x = \infty$, where $E(x)$ is a matrix of entire functions not singular in the finite plane, and where $k_1, k_2, \cdots, k_n$ are integers.*

For the proof of this lemma the reader is referred to the Mathematische Annalen, vol. 74 (1913), pp. 122–133. With its aid we may at once prove the theorem which follows:

Theorem IV. *Any matrix $F(x)$ with a finite number of isolated singularities*

(essential or non-essential) at the distinct points $x_1, x_2, \cdots, x_m, x_{m+1} = \infty$, *and no others, can be written as a product of factors* $E(x) W_m(x) W_{m-1}(x) \cdots W_1(x)$ *where these factors* W *are*

$$W_l(x) = E_l\left(\frac{1}{x - x_l}\right)[(x - x_l)^{k_j^{(l)}} \delta_{ij}].$$

Here $E(y), E_m(y), \cdots, E_1(y)$ *are matrices of entire functions not singular in the finite plane and* $k_j^{(l)}$ $(l = 1, 2, \cdots, m; j = 1, 2, \cdots, n)$ *are integers.*

Proof. It is clear that $F[(x_1 + 1/\xi)]$ has an isolated essential singularity at $\xi = \infty$ and so, by the above lemma, may be written as

$$A_1(\xi) E_1(\xi) (\xi^{k_j^{(1)}} \delta_{ij}),$$

where $E_1(\xi)$ is a matrix of entire functions which is not singular in the finite plane. Transferring back to the variable x, we obtain the decomposition

$$F(x) = F_1(x) E_1\left(\frac{1}{x - x_1}\right)[(x - x_1)^{-k_j^{(1)}} \delta_{ij}],$$

where $F_1(x)$ is not singular at $x = x_1$. It follows that $F_1(x)$ has singularities only at $x_2, \cdots, x_m$, and $x_{m+1} = \infty$.

By the same process we find

$$F_1(x) = F_2(x) E_2\left(\frac{1}{x - x_2}\right)[(x - x_2)^{-k_j^{(2)}} \delta_{ij}],$$

where $F_2(x)$ is not singular save at $x_3, \cdots, x_m$, and $x_{m+1} = \infty$.

Proceeding successively in this manner we obtain a set of similar equations by introducing matrices $F_3, \cdots, F_m$. The last equation is

$$F_{m-1}(x) = F_m(x) E_m\left(\frac{1}{x - x_m}\right)[(x - x_m)^{-k_j^{(m)}} \delta_{ij}],$$

where $F_m(x)$ is not singular save at infinity, and hence is a matrix $E(x)$ of entire functions of the desired type.

From this succession of equations we get the stated decomposition at once. It is not difficult to infer that the above decomposition is essentially unique.

7. The corresponding equivalence problem

We can now pass at once to the solution of the corresponding problem:

Equivalence problem.* *Given matrices* $G_1(x), \cdots, G_m(x)$ *each with one isolated singularity at the distinct points* $x_1, \cdots, x_m$ *respectively in the finite plane and with no other singular points; to construct a matrix* $F(x)$ *equivalent to*

*Cf. Proceedings of the American Academy of Arts and Sciences, vol. 49 (1913), pp. 521–568; in particular pp. 540–541.

matrices $G_1(x), \cdots, G_m(x)$ *at* $x_1, \cdots, x_m$ *respectively, and otherwise not singular save for a possible pole at infinity.*

Proof. We follow the method used in §§ 2, 5 in solving the similar problem for matrices of polynomials and entire functions.

Suppose that $T(x)$ has an isolated singularity at $x = \xi$. Let $[T(x)]_\xi$ be used to designate the matrix

$$E\left(\frac{1}{x-\xi}\right)[(x-\xi)^{-k_j}\delta_{ij}],$$

to which, in accordance with the lemma of the preceding section, $F(x)$ is equivalent at $x = \xi$. This matrix has no other singularity save for a possible pole at $x = \infty$. Here $E(y)$ is an entire function of y and not singular in the finite plane.

If now we write

$$H_1(x) = [G_1(x)]_{x_1} = G_1(x),$$

$$H_2(x) = [G_2(x)H_1^{-1}(x)]_{x_2},$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$H_m(x) = [G_m(x)H_1^{-1}(x)\cdots H_{m-1}^{-1}(x)]_{x_m},$$

we infer at once that the matrix $F(x)$ defined by

$$F(x) = H_m(x)\cdots H_1(x)$$

has the stated properties (see § 2).

The most general solution of the equivalence problem is $Q(x)F(x)$, *where* $Q(x)$ *is any polynomial matrix of constant determinant.*

8. The matrix with general singularities

The factorization theorems of §§ 1, 3, admit of a further extension as follows:

Theorem V. *Let* $x_1, x_2, \cdots$ *be an isolated set of points in the finite complex plane with a set of limit points* $[c]$, *one of which lies at infinity in the plane. Further, let* $\bar{E}_1(y), \bar{E}_2(y), \cdots$ *be arbitrary matrices of functions of the form* $E(y)(y^{k_j}\delta_{ij})$ *where* $E(y)$ *is a matrix of entire functions not singular in the finite plane, and* $k_1, \cdots, k_n$ *are integers. Then there exists a matrix* $G(x)$ *of analytic functions given by a formula*

$$G(x) = \lim_{m=\infty} Q_m(x)\bar{E}_m\left(\frac{1}{x-x_m}\right)\cdots\bar{E}_1\left(\frac{1}{x-x_1}\right),$$

where $Q_m(x)$ *is not singular in the finite plane, such that* $G(x)$ *is equivalent to* $\bar{E}_m[1/(x-x_1)]\cdots\bar{E}_1[1/(x-x_2)]$ *at* x_m $(m = 1, 2, \cdots)$ *and is otherwise not singular save at the points of* $[c]$.

Proof. We shall follow closely the method used in proving the analogous Theorem II (§ 3).

For convergence of a formula for $G(x)$ of the type indicated, it is sufficient to establish that $Q_{m+1}(x)$ can be so chosen that, in a region outside of an arbitrary small distance of x_{m+1} and within a circle of arbitrarily large radius with center at this point in the complex plane, the elements of

$$Q_{m+1}(x)\,\bar{E}_{m+1}\left(\frac{1}{x-x_m}\right)Q_m^{-1}(x) \tag{9}$$

can be made to differ in absolute value from the corresponding elements of the unit matrix by less than an arbitrarily assigned positive quantity.

But this is readily done as follows. The matrix

$$Q_m(x)\,\bar{E}_{m+1}^{-1}\left(\frac{1}{x-x_{m+1}}\right)$$

is obviously not singular except at x_{m+1} and at infinity. Hence by the lemma of § 3 we can find a matrix $Q_{m+1}(x)$, not singular in the finite plane, whose elements differ from those of the above matrix by less than an arbitrarily assigned quantity for

$$r \leqq |x - x_{m+1}| \leqq R,$$

r, R being arbitrary positive quantities.

Consequently we may make the elements of the matrix (9) differ from those of the unit matrix by less than an arbitrary positive quantity in this region for r as small and R as large as desired.

It is therefore clear that $Q_1(x)$, $Q_2(x)$, $\cdots$ may be taken in succession so that the limit in the theorem exists uniformly so long as x lies within any circle of arbitrarily large radius with center at the origin and not within an arbitrarily small distance of any point of $[c]$.

The matrix $G(x)$ thus defined has clearly the desired property. On the basis of this theorem we are led at once (as in § 5) to a solution of the problem:

EQUIVALENCE PROBLEM. *Given an infinite number of points $x_1, x_2, \cdots$ in the finite plane with a set of limit points $[c]$, one of which lies at infinity, and matrices $\bar{E}_1(y), \bar{E}_2(y), \cdots$ of the form $E(y)(y^{k_{ij}}\delta_{ij})$ where $E(y)$ is a matrix of entire functions, not singular in the finite plane, and $k_1, \cdots, k_n$ are integers; to construct a matrix $G(x)$ equivalent to $\bar{E}_1[1/(x-x_1)]$, $\bar{E}_2[1/(x-x_2)]$, $\cdots$ at $x_1, x_2, \cdots$ respectively, and otherwise not singular save at the points of $[c]$.*

The most general solution of this equivalence problem is $M(x)G(x)$ where $M(x)$ is an arbitrary matrix of analytic functions whose singularities all fall at the points of $[c]$.

9. An application

I shall give now a simple application of the preceding theorems. This application may be stated as follows: *Let* $e(x)$ *be an entire function and let* $e_0(x)$, $e_1(x)$, $e_2(x)$, $e_3(x)$, $\cdots$ *denote the functions* x, $e(x)$, $e[e(x)]$, $e\{e[e(x)]\}$, $\cdots$ *respectively. Furthermore let* $F(x)$ *denote an arbitrary matrix of single-valued analytic functions. A matrix* $G(x)$ *of single-valued analytic functions can then be found whose singular points fall at those of* $F[e_k(x)]$ $(k = 0, 1, 2, \cdots)$ *and at the limit points* $[c]$ *of this set, and such that* $G(x)$ *is equivalent to* $G[e(x)]F(x)$ *save at these limit points.*

Let us write down for reference the symbolic infinite product to the left

$$\cdots F[e_2(x)]F[e_1(x)]F[e_0(x)].$$

The singular points of any factor $F[e_n(x)]$ not belonging to $[c]$ are isolated singular points.

Consider now any such singular point $\bar{x}$ not a point of $[c]$. It is possible to take a positive integer k so large that all the factors before the kth in the above product are not singular at $\bar{x}$ nor in its neighborhood, since $\bar{x}$ is not a point of $[c]$.

We shall demand of a matrix $G(x)$ that it be equivalent to the last k factors of the product

$$A[e_{k-1}(x)]\cdots A[e_0(x)]$$

at every point $\bar{x}$.

It is clear that if k be increased $G(x)$ will still have the same property since the factors introduced are not singular at $\bar{x}$, and appear on the left of the previous factors.

The set of such points $\bar{x}$ is clearly numerable. In fact any set of isolated points is numerable.

According to § 8 a matrix $G(x)$ may be found which satisfies these relations of equivalence at the singular points $\bar{x}$ of $F[e_m(x)]$ $(m = 0, 1, \cdots)$ and which has no further singular points save the points of $[c]$.

This matrix $G(x)$ will have the desired property that $G(x)$ is equivalent to $G[e(x)]F(x)$ save at the points of $[c]$.

In proving this statement it is first to be observed that if $\bar{x}$ is a singular point of $F[e_m(x)]$, and if a value $\bar{y}$ exists such that $\bar{x} = e(\bar{y})$, then $\bar{y}$ is a singular point of $F[e_{m+1}(y)]$. For we have identically

$$F[e_m(x)] = F[e_{m+1}(y)],$$

where $x = e(y)$. Hence, if the matrix on the right is not singular at $\bar{y}$, the elements of the equal matrix on the left are analytic in x at $\bar{x}$, since they can not be multiple-valued functions of x. Moreover, since as x approaches $\bar{x}$, the variable y approaches $\bar{y}$, the determinant of the right-hand side will not

approach zero in this case. Thus $F[e_m(x)]$ would not be singular at $\bar{x}$, contrary to hypothesis.

The converse statement is obviously true, i. e., if $F[e_{m+1}(y)]$ is singular at $\bar{y}$ and if $\bar{x} = e(\bar{y})$, then $F[e_m(x)]$ is singular at $\bar{x}$.

Thus we derive the precise totality of singular points of successive factors of the symbolic infinite product from the singularities $\bar{x}_0$ of the first factor $F(x)$ by forming $e_{-1}(\bar{x}_0)$, $e_{-2}(\bar{x}_0)$, $\cdots$ where $e_{-1}(x)$, $e_{-2}(x)$, $\cdots$ denote the inverse functions to $e_1(x)$, $e_2(x)$, $\cdots$ respectively, which are of course in general multiple valued.

Consider now any point not belonging to $[e]$.

If this point ξ is not a singular point of some matrix $F[e_m(x)]$ then $G(x)$ is certainly not singular at ξ, as has been stated. Moreover $e(\xi)$ is not a singular point for $G(x)$, inasmuch as in the contrary case some $F[e_m(x)]$ is singular at $e(\xi)$ and thus the corresponding $F[e_{m+1}(x)]$ is singular at ξ. Consequently $G[e(x)]$ is not singular at ξ. It follows that $G(x)$ is equivalent to $G[e(x)]F(x)$ at any such point ξ.

It remains to consider a point $\bar{x}$ for which some $F[e_m(x)]$ is singular. At such a point $G(x)$ has been chosen to be equivalent to a product

$$F[e_k(x)]F[e_{k-1}(x)]\cdots F[e_0(x)],$$

where k has been taken so large as to include all the factors singular at $\bar{x}$. The point $e(\bar{x})$ will be a singular point for $F[e_{m-1}(x)]$ unless m is zero. Putting aside this contingency for the moment, we see that $G(x)$ is equivalent to

$$F[e_{k-1}(x)]F[e_{k-2}(x)]\cdots F[e_0(x)]$$

at $e(\bar{x})$. Consequently $G[e(x)]$ is equivalent at $\bar{x}$ to

$$F[e_k(x)]F[e_{k-1}(x)]\cdots F[e_1(x)].$$

Hence $G(x)$ is equivalent to $G[e(x)]F(x)$ at such a point $\bar{x}$.

If, on the other hand, m is zero then the integer k may be taken zero. In this case $G(x)$ is equivalent to $F(x)$ at $\bar{x}$, while $G(x)$ is not singular at $e(\bar{x})$. In this case also $G(x)$ is equivalent to $G[e(x)]F(x)$ at $\bar{x}$.

The italicized statement made is now proved.

The usual existence theorems for linear difference and q-difference equations show that if $e(x) = x + 1$ or qx respectively, and if $F(x)$ is a polynomial matrix we may choose $G(x)$ *equal to* $G[e(x)]F(x)$.

HARVARD UNIVERSITY
March, 1916